STOCHASTIC STABILITY OF REPLICATOR DYNAMICS WITH RANDOM JUMPS

BY

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DISSERTATION

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Abstract

We further generalize the stochastic version of the replicator dynamics due to Fudenberg and Harris [7]. In particular, we add a random jump term to the payoff function to simulate anomalous events and their effects on the fitness. Assuming a 2×2 game and using a particular characteristic of the jump functions we are able to estimate the ergodic measure for all games. Lastly, working with results and methods developed by Imhoff [11], we prove some stability theorems for an arbitrary $n \times n$ game.

I dedicate this work to my wife and daughter. With hard work and determination my "pooky bear", you are able to accomplish any goal.

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Chapter 1

Introduction

1.1 Motivation

Game theory is a way to put a mathematical structure on the strategic interplay of agents in a competitive environment. Evolutionary game theory is the study of how the game evolves over time. To structure the game we need to identify possible tactics and give a quantitative identification to distinguish the differences between each one. The possible tactics that the players use we will call **strategies**, which will be labeled S_1 through S_n, for some fixed $n \in \mathbb{N}$. Furthermore, the payoff of a player using S_i against a player using S_j will be denoted as a_{ij}. Moreover, for the payoffs we will assume that only the strategy matters, not who is playing the strategy. This is called a symmetric game. Furthermore, a player will only play against one other player at a time. We call $S^* = (S_{i_1}, S_{i_2})$ a **Nash equilibrium** only if the best set of strategies to play against S^* is itself. Hence, if the second player chooses any other strategy than S_{i_2}, they will not receive a better payoff.

Consider an environment where within a population there are n subpopulations such that a player in the $\mathbf{i^{th}}$ **subpopulation** is employing S_i and each subpopulation contains players where no individual player has the ability to affect any subpopulation. This assumption is a nice approximation to a large population in which an individual's choice of a strategy does not affect the proportions of the strategies being played. Moreover, the space being worked in is connected and not discrete. The game will be played in the following way: an individual in each subpopulation will be matched against another individual from a random subpopulation. The game will evolve by having this process independently repeated over time.

Classic replicator dynamics focuses on the adjustment process in which the proportion of the population playing a given strategy grows at a rate proportional to the current *"fitness"* of that subpopulation minus the average fitness of all of the distinct subpopulations. Thus, growth is dependent on how the fitness of each subpopulation does against *"nature"*. Given this deterministic model, we have that all Nash equilibria are fixed points of the dynamic, and all strict Nash equilibria are asymptotically stable fixed points. Fudenberg and Harris [7] noted that due to random fluctuations of the payoffs for the strategies, this deterministic model was not quite correct. To adjust for this the authors proceeded to add a Brownian term to the *"payoff functions"*. The authors then take a 2×2 symmetric game (two strategies), a linear payoff function, and give conditions as to which of the two subpopulations the process will settle to, i.e., as to whether the proportion of the 1^{st} subpopulation will converge to 0 or 1 as $t \to \infty$, and with what probability. (There is no need to do any analysis on the 2^{nd} subpopulation since it is the complement of the 1^{st}.) Determining the stochastic stability of the evolutionary process will give us an idea of how the population will evolve over a long period of time.

Taking the ideas of Fudenberg and Harris [7], we conjecture that the Brownian term is not enough to fully capture the effects of random fluctuations of the payoffs and so we will add a random jump term, along with the Brownian term, to the payoff functions. This jump term will be driven by a Levy process which we assume is independent of each of the Wiener processes. Our main goal is to give an analogous results to that of Fudenberg and Harris [7] and compare our results to that of classical results.

1.2 Classical Game Theory

We will only consider a game with a 1^{st} and 2^{nd} subpopulation, however, most of the derivations will be done in generality. The possible interactions and payoffs will be written as

	S_1	S_2
S_1	a_{11}, a_{11}	a_{12}, a_{21}
S_2	a_{21}, a_{12}	a_{22}, a_{22}

,

where the first entry is the payoff to the "row player" and the second is the payoff to the "column player". To make the notation above more tangible we will go over some examples.

Example 1.2.1. *Consider the game*

	S_1	S_2
S_1	$5, 5$	$3, 2$
S_2	$2, 3$	$1, 1$

.

This is called a strategy 1 dominate game, since the best payoff for both players is when they both play S_1. Hence (S_1, S_1) is a Nash equilibrium.

Example 1.2.2. *A game of the type*

	S_1	S_2
S_1	$3, 3$	$0, 0$
S_2	$0, 0$	$2, 2$

is called a coordination game. The strategies (S_1, S_1) and (S_2, S_2) are both Nash equilibria. Even though the better payoff is with (S_1, S_1), if the other player is playing S_2, it would be in your best interest to play S_2 as well. There is *also another Nash equilibrium at $(2/5S_1 + 3/5S_2, 2/5S_1 + 3/5S_2)$. The way to interpret the last strategy is a player would play S_1 2/5 the time and S_2 3/5 of the time.*

Example 1.2.3. *A game of the type*

	S_1	S_2
S_1	$-10, -10$	$10, 1$
S_2	$1, 10$	$-1, -1$

is known as a mixed strategy game. The strategies (S_1, S_2) and (S_2, S_1) are Nash equilibria. Furthermore, the strategy $(1/2S_1 + 1/2S_2, 1/2S_1 + 1/2S_2)$, is also a Nash equilibrium.

Remark 1.2.1. *To find the mixed Nash equilibria, take p as the probability that the row player uses S_1 and q as the probability that the column player uses S_1. So the expected payoff for any player would be $a_{11}pq + a_{22}(1-p)(1-q) + a_{12}p(1-q) + a_{21}(1-p)q := l(p,q)$. Solving $\dfrac{\partial l(p,q)}{\partial p} = 0$ would give us these values.*

Example 1.2.4. *Finally, we will look a game which is called prisoner's dilemma,*

	S_1	S_2
S_1	$4, 4$	$-2, 7$
S_2	$7, -2$	$0, 0$

This is actually a strategy 2 dominate game. For if both players are using S_1, then there is incentive for one player to use S_2. Now the payoff for the player that stayed with S_1 is getting a negative payoff and thus has incentive to use S_2 as well. The strategy pair (S_1, S_1) is known as Pareto optimal, and as just seen, is not always a Nash equilibria.

The examples given above are that of a game played once. This begs the natural question: how would the population evolve if the game is played repeatedly over time? Let $r_i(t)$ be the size of the i^{th} subpopulation at time t, $\mathbf{r}(t) := (r_1(t), \ldots, r_n(t))^T$, $R(t) := \sum_i r_i(t)$ and $\mathbf{s}(t) := (s_1(t), \ldots, s_n(t))^T$ where $s_i(t) := r_i(t)/R(t)$ and $s_i(0)$ the initial condition. So $s_i(t)$ is the proportion of the population playing S_i at time t. Define $u(\cdot,\cdot)$ to be the **payoff function**. By our construction $u(s_i(t), s_j(t)) = a_{ij}$, but at times we will talk about the payoff function in generality. Finally, define the **fitness function** u_i of the i^{th} subpopulation as the expected payoff, i.e., $u_i(\mathbf{r}(t)) = \sum_j u\left(\frac{r_i(t)}{R(t)}, \frac{r_j(t)}{R(t)}\right)\frac{r_j(t)}{R(t)} \left(= \sum_j u(s_i(t), s_j(t))s_j(t) = u_i(\mathbf{s}(t))\right)$. Taking $A = (a_{ij})$ to be an $n \times n$ matrix with the entries being the payoffs, we have that $u_i(\mathbf{r}(t)) = \left(A\left(\frac{r_1(t)}{R(t)}, \ldots, \frac{r_n(t)}{R(t)}\right)^T\right)_i$. We will call A the **payoff matrix**.

We will assume that the deterministic growth model for the i^{th} subpopulation (for all i) is of the form

$$\dot{r}_i(t) = r_i(t)u_i(\mathbf{r}(t)).$$

It should be noted that this growth is similar to the Lotka-Volterra model. Normalization reveals

$$\dot{s}_i(t) = \sum_{j \neq i} s_i(t)s_j(t)[u_i(\mathbf{s}(t)) - u_j(\mathbf{s}(t))] = s_i(t)\left[u_i(\mathbf{s}(t)) - \sum_j s_j(t)u_j(\mathbf{s}(t))\right], \tag{1.1}$$

which are replicator dynamics.

Remark 1.2.2. *It is not obvious that the dynamics above equals replicator dynamics. However, we have that*

$$\dot{s}_i = \sum_{j \neq i} s_i s_j [u_i - u_j] = s_i\left[u_i \sum_{j \neq i} s_j - \sum_{j \neq i} u_j s_j\right] = s_i\left[u_i \sum_j s_j - \sum_j u_j s_j\right],$$

and since

$$\begin{aligned}
\dot{\left(\sum_i s_i\right)} &= \sum_i \dot{s}_i = \sum_i \sum_{j \neq i} s_i s_j [u_i - u_j] \\
&= \sum_i \sum_{j \neq i} s_j s_i u_i - \sum_i \sum_{j \neq i} s_i s_j u_j = 0,
\end{aligned}$$

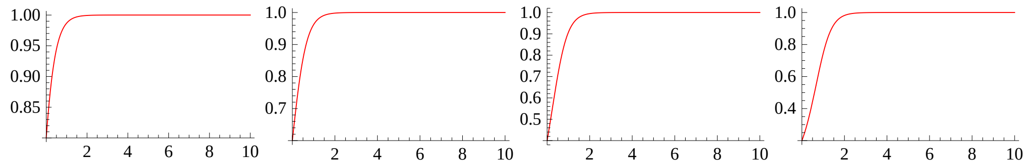
coupled with the initial condition $\sum_i s_i(0) = 1$ *gives* $\dot{s}_i = s_i\left[u_i - \sum_j u_j s_j\right]$. *We have further shown that the PDE stays on the simplex.*

Since our population is only the 1^{st} and 2^{nd} subpopulations, using the fact that $s_2(t) = 1 - s_1(t)$, we will simplify the replicator equation. So

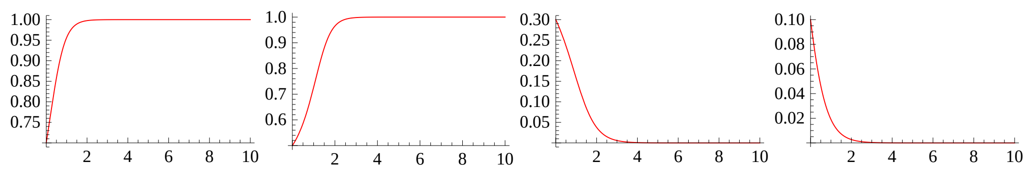
$$\dot{s}_1(t) = s_1(t)(1 - s_1(t))[(a_{12} - a_{22}) + (a_{11} - a_{21} + a_{22} - a_{12})s_1(t)]. \tag{1.2}$$

In this dynamic (and the general replicator equation), Nash equilibria are fixed points, but are not necessarily stable [10]. We will now take the examples of games given above and show how the population evolves.

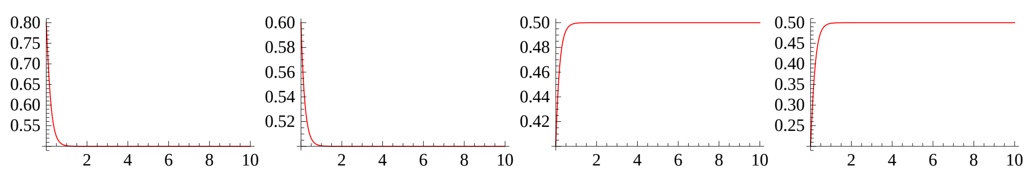
Example 1.2.5. *The dynamics of the strategy 1 dominate game has the form of* $\dot{s}_1(t) = s_1(t)(1 - s_1(t))[2 + s_1(t)]$*. Clearly* 0 *and* 1 *are fixed points with* 0 *unstable and* 1 *stable, since the right side is positive away from 0 and 1. Thus, over a period of time, the entire population will be playing the* S_1 *strategy.*



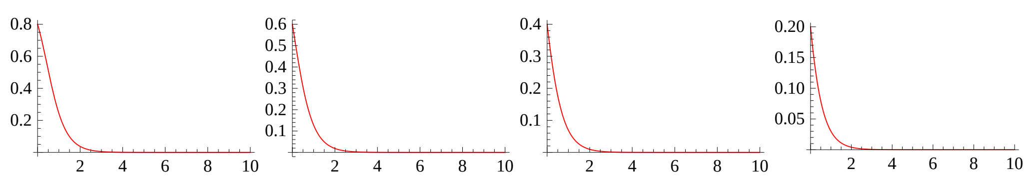
Example 1.2.6. *The* $\dot{s}_1(t) = s_1(t)(1 - s_1(t))[-2 + 5s_1(t)]$ *represents the coordination game, where* 0 *and* 1 *are clearly stable. In particular, if the initial condition is greater than* $2/5$*, the flow is towards* 1 *since the right hand side is positive, and if less than* $2/5$ *the flow is to* 0 *since the right hand side is negative. Thus, the fixed point* $2/5$ *is unstable.*



Example 1.2.7. *For example (1.2.3), we see that* $\dot{s}_1(t) = s_1(t)(1 - s_1(t))[11 - 22s_1(t)]$ *has the fixed points* 0 *and* 1 *which are unstable, while* $1/2$ *is stable. This can be seen by noticing that the right hand side is positive on the interval* $(0, 1/2)$*, and negative on the interval* $(1/2, 1)$*.*



Example 1.2.8. *Finally, we see that* $\dot{s}_1(t) = s_1(t)(1 - s_1(t))[-2 - s_1(t)]$ *has the fixed points* 0 *and* 1*, with* 1 *unstable and* 0 *stable, considering that the right hand side is negative away from 0 and 1.*



The dynamics of all of the examples, except for (1.2.7), reinforce how we would expect the population to evolve over time. For example (1.2.7), we would expect the population to converge to one of the pure strategy Nash equilibria, but since this is a one population model, the mixed strategy is the closet representation of this. The two subpopulation replicator dynamic does give us those stable fixed points.

1.3 Exploring the Fudenberg and Harris Model

We will derive the model of Fudenberg and Harris [7], following their procedure, and keeping the notation of §1.2. Fudenberg and Harris [7] add a Brownian motion to the payoff function (this random term is considered to be a "stochastic shock") to change $\dot{r}(t)$ to the stochastic differential equation (SDE)

$$dr_i(t) = r_i(t)\Big(u_i(\mathbf{r}(t))dt + \sigma_i dW_i(t)\Big) \ \forall i, \tag{1.3}$$

where $\sigma_i \in \mathbb{R}_+$ and, W_i and W_j, for $i \neq j$, are independent. Applying Ito's lemma to $s_i(t)$ gives the SDE

$$ds_i(t) = \sum_{j \neq i} s_i(t)s_j(t)\bigg[\Big(u_i(\mathbf{s}(t)) - u_j(\mathbf{s}(t))\Big)dt + \Big(\sigma_j^2 s_j(t) - \sigma_i^2 s_i(t)\Big)dt + \Big(\sigma_i dW_i(t) - \sigma_j dW_j(t)\Big)\bigg]. \tag{1.4}$$

Though we will skip the derivation, it can be seen how this equality comes about through the details of the construction of our new model.

We have that for all i, s_i is strictly in the simplex. We will bypass the proof since this is shown later for our new model and the details are similar. Recalling that the authors take the two subpopulation model, we have that $s_2(t) = 1 - s_1(t)$. Now define $W(t) := \Big(\sigma_1 W_1(t) - \sigma_2 W_2(t)\Big)\Big/\sigma$, where $\sigma := \sqrt{\sigma_1^2 + \sigma_2^2}$. Note that W is a standard Wiener process (this can be seen from characteristic functions and the independence of W_1 and W_2). Lastly, since $u_1(s(t)) = a_{11}s_1(t) + a_{12}s_2(t)$ and $u_2(s(t)) = a_{22}s_2(t) + a_{21}s_1(t)$, equation (1.4) yields the SDE

$$\begin{aligned} ds_1(t) &= s_1(t)\Big(1 - s_1(t)\Big)\bigg[\Big\{(a_{11} - a_{21})s_1(t) - \big(a_{22} - a_{12}\big)\big(1 - s_1(t)\big) + \big(1 - s_1(t)\big)\sigma_2^2 - s_1(t)\sigma_1^2\Big\}dt + \sigma\tilde{W}(t)\bigg] \\ &= s_1(t)\Big(1 - s_1(t)\Big)\bigg[a_{12} - a_{22} + \sigma_2^2 + \Big\{a_{11} - a_{21} - \sigma_1^2 + a_{22} - a_{12} - \sigma_2^2\Big\}s_1(t)\bigg]dt \\ &+ \sigma s_1(t)\Big(1 - s_1(t)\Big)dW(t). \end{aligned} \tag{1.5}$$

This brings the analysis to a one dimensional process, in which there are a lot of results to utilize. In particular, using results from Gihman and Skorohod ([8], page 119), Fudenberg and Harris [7] give conditions as to whether $s_1(t) \to 0$ or $s_1(t) \to 1$ as $t \to \infty$. We will now briefly explain the theorem that was applied. Note that the statement of the original theorem is over the real line, however the proof for a finite interval is very similar. Let $dx(t) = \alpha(x(t))dt + \beta(x(t))dW(t)$, where W is a standard Wiener process, α and β are locally Lipschitz, $|\alpha(x)| + |\beta(x)| \leq K(1 + |x|)$, for some $K > 0$, and β is a strictly positive function. Also, take $x(0)$ to be the nonrandom initial position and an arbitrary $z \in (0, 1)$. Now define

$$I_1 = \int_0^{x(0)} \exp\left[-\int_z^x [2\alpha(y)/\beta^2(y)]dy\right] dx,$$

and

$$I_2 = \int_{x(0)}^{1} \exp\left[-\int_z^x [2\alpha(y)/\beta^2(y)]dy\right] dx.$$

The series of lemmas then tells us that: if I_1 is infinite and I_2 is finite then the process converges to 1 a.s.; if I_1 is finite and I_2 is infinite then the process converges to 0 a.s.; if I_1 and I_2 are both finite then the process converges to 1 with probability $\dfrac{I_1}{I_1+I_2}$ and to 0 with probability $\dfrac{I_2}{I_1+I_2}$; and if I_1 and I_2 are both infinite then the process oscillates forever with $P\left(\liminf_{t\to\infty} x(t) = 0\right) = P\left(\limsup_{t\to\infty} x(t) = 1\right) = 1$. We are now ready to state the author's conclusion.

Proposition 1.3.1. *(Fudenberg and Harris [7])*

1. *If* $a_{11} - a_{21} > (\sigma_1^2 - \sigma_2^2)/2$ *and* $a_{22} - a_{12} < (\sigma_2^2 - \sigma_1^2)/2$ *then* $s_1(t) \to 1$ *as* $t \to \infty$ *a.s.*

2. *If* $a_{11} - a_{21} < (\sigma_1^2 - \sigma_2^2)/2$ *and* $a_{22} - a_{12} > (\sigma_2^2 - \sigma_1^2)/2$*, we have* $s_1(t) \to 0$ *as* $t \to \infty$ *a.s.*

3. *If* $a_{11} - a_{21} > (\sigma_1^2 - \sigma_2^2)/2$ *and* $a_{22} - a_{12} > (\sigma_2^2 - \sigma_1^2)/2$ *then* $s_1(t) \to 1$ *as* $t \to \infty$ *with probability* $\frac{I_1}{I_1+I_2}$ *and* $s_1(t) \to 0$ *as* $t \to \infty$ *with probability* $\frac{I_2}{I_1+I_2}$.

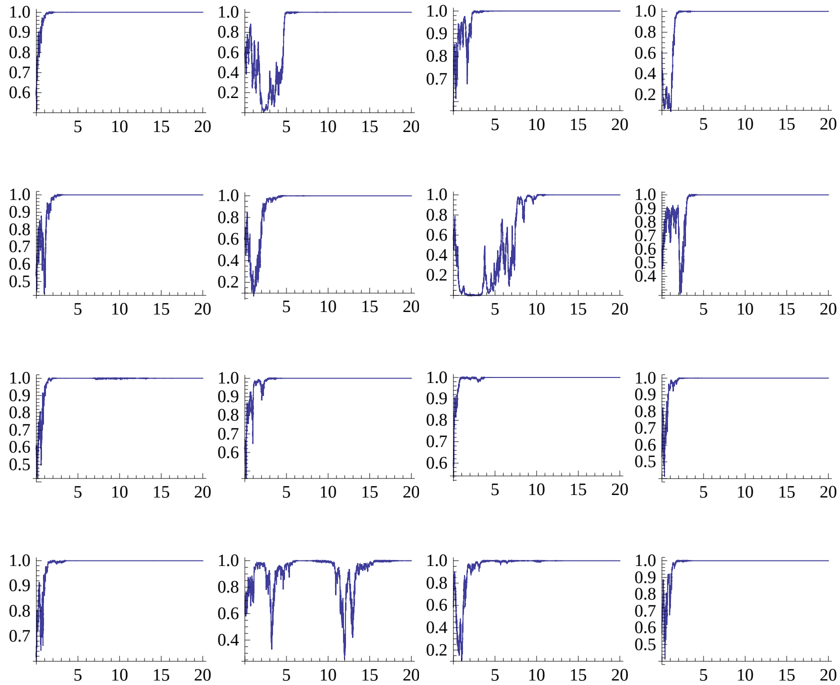
4. *If* $a_{11} - a_{21} < (\sigma_1^2 - \sigma_2^2)/2$ *and* $a_{22} - a_{12} < (\sigma_2^2 - \sigma_1^2)/2$ *then* $P\Big(\liminf_{t\to\infty} s_1(t) = 0\Big) = P\Big(\limsup_{t\to\infty} s_1(t) = 1\Big) = 1$.

   Furthermore, the distributions of the process $s_1(t)$ *converge to a unique ergodic distribution as* $t \to \infty$.

The first and second inequalities correspond to the strategy 1 dominate game and strategy 2 dominate game, respectively, and inequality 3 is similar to the coordination game. Inequality 4 relates to the mixed strategy game, however, a point mass does not exist in the this distribution. Since there is strictly positive variance in the unit interval, this should be the case.

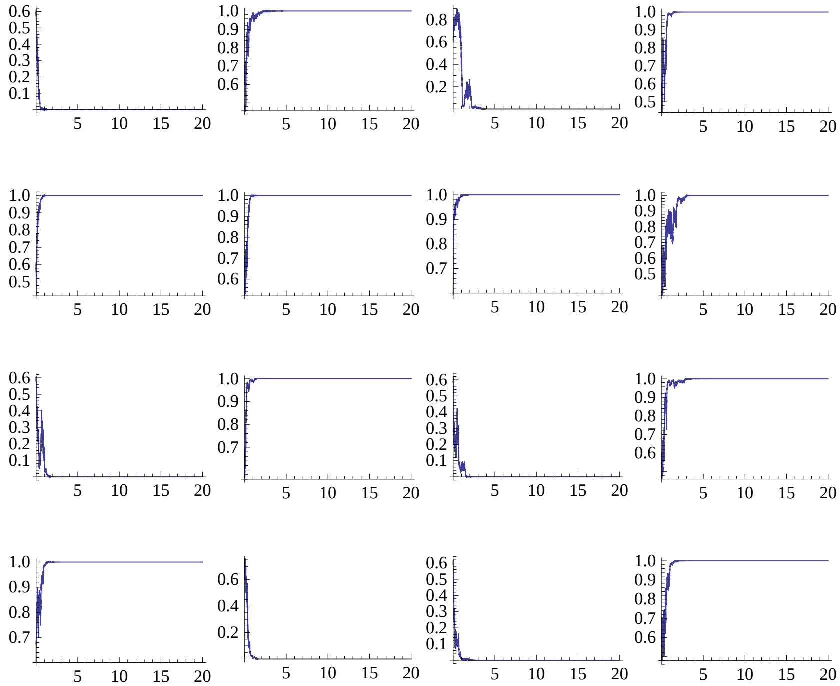
Now that we have the results of this model, we will reexamine the examples given in the previous sections to the stochastic model. Even though we are able to choose the size of the σ_i's in anyway we would, we will pick size so as to keep the inequalities as they are in the examples. Included in these examples are simulations of the respective SDE to reiterate what the theory says about the stochastic stability.

Example 1.3.1. *Taking $\sigma_1 = 2$ and $\sigma_2 = 1$ with the strategy 1 dominate example gives the SDE $ds_1(t) = s_1(t)(1 - s_1(t))[3 - 4s_1(t)]dt + \sqrt{5}s_1(t)(1 - s_1(t))dW(t)$.*



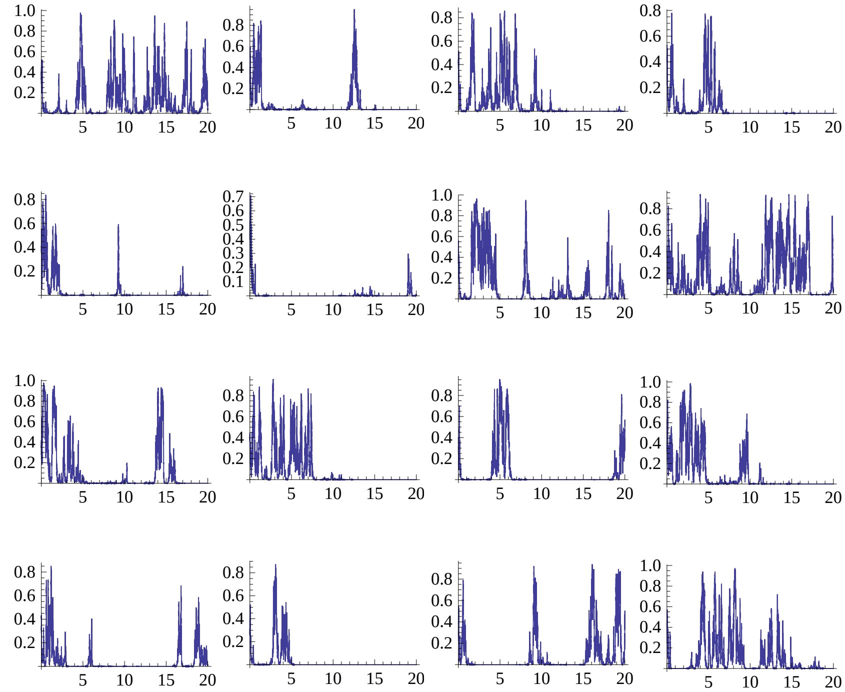
As we can see from the simulations, even though there is some fluctuations towards 0, in the long run the process flows towards 1, which corresponds exactly to the what was proved by Fudenberg and Harris in Proposition 1.3.1.

Example 1.3.2. *For $\sigma_1 = 2$ and $\sigma_2 = 1$ coupled with the coordination game example yields the $ds_1(t) = s_1(t)(1 - s_1(t))[-1 + 2s_1(t)]dt + \sqrt{5}s_1(t)(1 - s_1(t))dW(t)$ with initial condition $s(0) = .4$.*



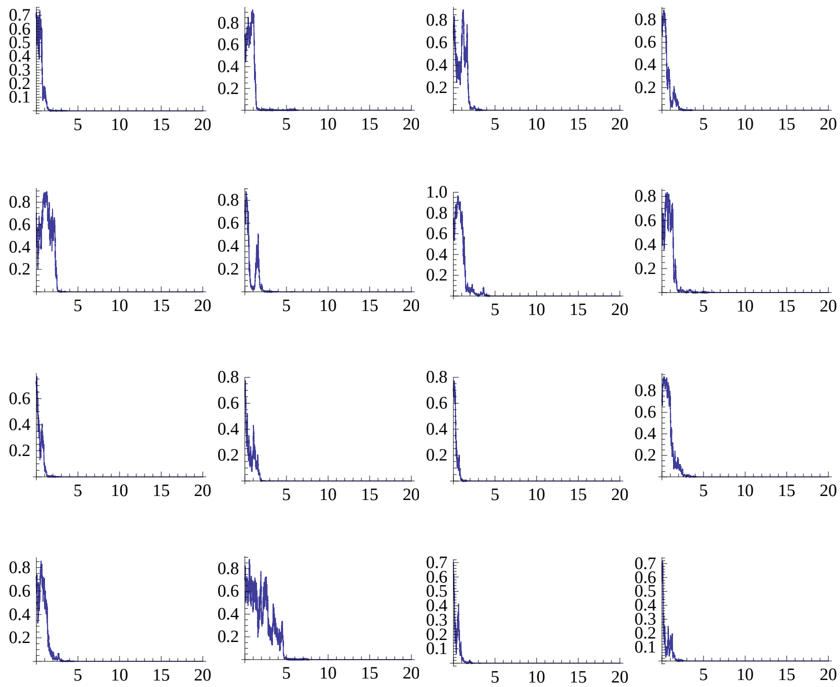
According to Propostion 1.3.1, the probability to converge to 0 is about .42, and the probability to converge to 1 is about .58, which very close the simulations.

Example 1.3.3. *The mixed strategy game with* $\sigma_1 = \sqrt{15}$ *and* $\sigma_2 = 4$*, tells us we have the SDE* $ds_1(t) = s_1(t)(1 - s_1(t))[17 - 54s_1(t)]dt + \sqrt{31}s_1(t)(1 - s_1(t))dW(t)$.



From the consistent spikes in the simulation we conclude that the process is recurrent and hence coincides to what was shown.

Example 1.3.4. *Example (1.4) with the sigma values, $\sigma_1 = 1$ and $\sigma_2 = 2$, is the SDE $ds_1(t) = s_1(t)(1 - s_1(t))[-4s_1(t)]dt + \sqrt{5}s_1(t)(1 - s_1(t))dW(t)$.*



The flow towards 0 validates what was shown by Fudenberg and Harris.

1.4 Replicator Dynamics with Deterministic Jumps

Following the method of Fudenberg and Harris [7], we will add a step function into the payoff function and show how this term affects the replicator dynamics. Take $r_i(t)$ $(i = 1, 2)$ to evolve in the following way,

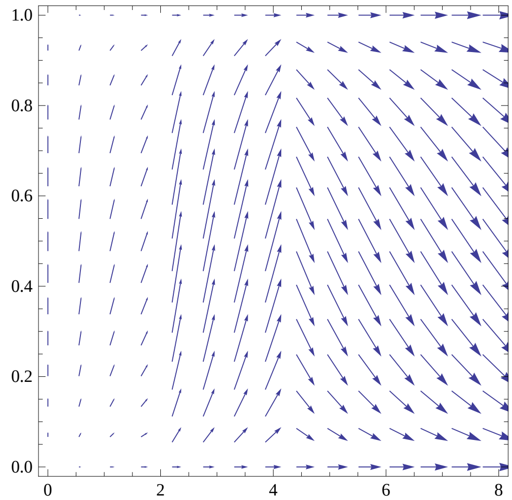
$$\dot{r}_i(t) = r_i(t)(u_i(\mathbf{r}(t)) + h_i(t)),$$

where $h_i(t)$ is a step function. Using remark (1.2) and equation (1.2), our replicator equation looks like

$$\begin{aligned}\dot{s}_1(t) &= s_1(t)(1 - s_1(t))[(a_{12} - a_{22}) + (a_{11} - a_{21} + a_{22} - a_{12})s_1(t) + h_1(t) - h_2(t)]\\ &= s_1(t)(1 - s_1(t))\times\\ &\times [\{a_{12} - a_{22} + h_1(t) - h_2(t)\} + (\{a_{11} - a_{21} + h_1(t) - h_2(t)\} + \{a_{22} - a_{12} - h_1(t) + h_2(t)\})s_1(t)].\end{aligned}$$

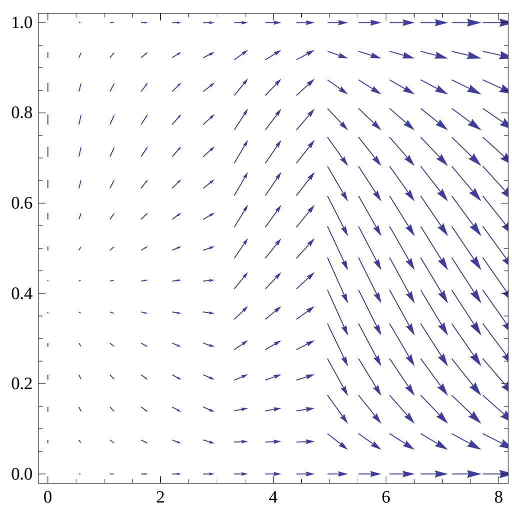
Thus, the jump terms affect the payoffs, and if large enough, change the fixed points and their stability. We will take the examples of §1.2 and show how their dynamics change.

Example 1.4.1. *Take* $h_1(t) = 5\chi_{[2,\infty)}(t)$*,* $h_2(t) = 13\chi_{[4,\infty)}(t)$ *and the strategy 1 dominate game. So* $\dot{s}_1(t) = s_1(t)(1 - s_1(t))[2 + s_1(t) + h_1(t) - h_2(t)]$.



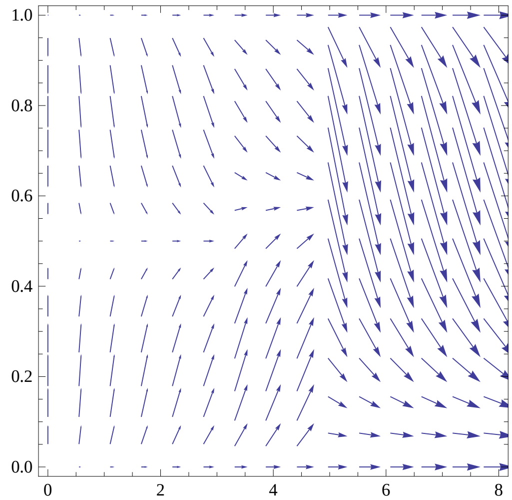
The flow is towards 1 and when the first jump comes, this flow to 1 is intensified, however, after the second jump comes, this flow is switched to 0.

Example 1.4.2. *Take* $h_1(t) = 2\chi_{[3,\infty)}(t)$*,* $h_2(t) = 8\chi_{[5,\infty)}(t)$ *and the coordination game. So* $\dot{s}_1(t) = s_1(t)(1 - s_1(t))[-2 + 5s_1(t) + h_1(t) - h_2(t)]$.



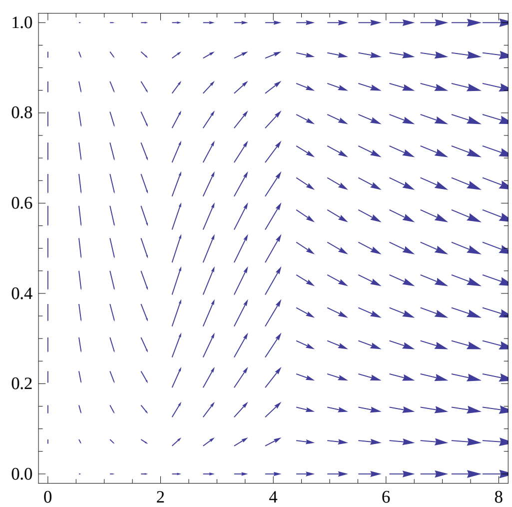
The flow is towards 0 *or* 1*, depending on whether the initial condition is below or above* 2/5 *respectively. When the first jump makes an impact, the flow changes to* 1*, and after the second jump hits, the flow changes direction to* 0.

Example 1.4.3. *Take* $h_1(t) = 2\chi_{[3,\infty)}(t)$*,* $h_2(t) = 13\chi_{[5,\infty)}(t)$ *and the mixed strategy game. So* $\dot{s}_1(t) = s_1(t)(1 - s_1(t))[11 - 22s_1(t) + h_1(t) - h_2(t)]$*.*



One can see that $1/2$ *is the stable point, and after the first jump, the stability moves to* $13/22$*. After the second jump, there is a significant shift so that the flow is now to* 0*.*

Example 1.4.4. *Take* $h_1(t) = 6\chi_{[2,\infty)}(t)$*,* $h_2(t) = 5\chi_{[4,\infty)}(t)$ *and prisoner's dilemma. So* $\dot{s}_1(t) = s_1(t)(1-s_1(t))[-2-s_1(t) + h_1(t) - h_2(t)]$*.*



The flow is towards 0 *and after the first jump the flow is switched to* 1*. The second jump switches the flow back to* 0*, however, not as intense as before the jumps.*

Chapter 2

The Model for Replicator Dynamics with Jumps

2.1 An Introduction to Poisson Integrals

In order to add a random jump term into the stochastic replicator dynamics, we need to understand the machinery behind this term. Take X to be an $\mathbb{R}^d$-valued Lévy process on $(\Omega, \mathcal{F}, P)$ and $B \in \mathcal{B}\big(\mathbb{R}^d \backslash \{0\}\big)$. Fixing $\omega \in \Omega$, define

$$\Delta X_\omega(s) := X_\omega(s) - X_\omega(s-)$$

and

$$N_\omega(t, B) := \#\Big\{0 \le s \le t : \Delta X_\omega(s) \in B\Big\} = \sum_{0 \le s \le t} 1_B\Big(\Delta X_\omega(s)\Big).$$

So for each $\omega \in \Omega$ and $t \in (0, \infty)$, the set function $B \to N_\omega(t, B)$ is a random counting measure. Moreover, $E\big[N_\omega(t, \cdot)\big] = \int_\Omega N_\omega(t, \cdot) dP(d\omega)$ is a Borel measure.

Definition 2.1.1. *([2]) We call $B \in \mathcal{B}\big(\mathbb{R}^d \backslash \{0\}\big)$* ***bounded below*** *if $0 \notin \overline{B}$.*

Lemma 2.1.1. *([2], page 87) If B is bounded below then $N_\omega(t, B) < \infty$ a.s. for all t.*

The randomness of this measure is a convient way to help us inject a noncontinuous "stochastic shock" into the payoff function. However, in order to implement it's application, we will need to exam this measure more closely. The theorem below gives some important properties of this counting measure. Before we begin the statement of the theorem we need to define the measure $\nu(\cdot) := E[N_\omega(1, \cdot)]$.

Theorem 2.1.1. *([2], page 88) If B is bounded below, then $\Big(N_\omega(t, B)\Big)_{t \in \mathbb{R}_+}$ is a Poisson process with* ***intensity*** *$\nu(B)$. Moreover, for $B_1, \ldots, B_n \in \mathcal{B}\big(\mathbb{R}^d \backslash \{0\}\big)$, where $B_i \cap B_j = \emptyset$ for $i \ne j$, $\Big(N_\omega(t, B_1)\Big)_{t \in \mathbb{R}_+}, \ldots, \Big(N_\omega(t, B_n)\Big)_{t \in \mathbb{R}_+}$ are independent processes.*

Note 2.1.1. *Since $N(t,\cdot)$ is generated by a Lévy process, $\nu(\cdot)$ is a Lévy measure [18], i.e., ν is a Borel measure with* $\displaystyle\int_{\mathbb{R}^d\setminus\{0\}}\Big(|x|^2\wedge 1\Big)\nu(dx)<\infty.$

Definition 2.1.2. *We call $N_\omega(t,\cdot)$ the **Poisson measure**, $\nu(\cdot)$ the **intensity measure** and $\tilde{N}_\omega(t,\cdot):=N_\omega(t,\cdot)-t\nu(\cdot)$ the **compensated Poisson measure**.*

The next claim gives us an important characteristic of the compensated Poisson measure and the reason why this measure is going to be the basis of our jump term instead of the just the Poisson measure.

Claim 2.1.1. *For B bounded below, $\tilde{N}_\omega(t,B)$ is a martingale measure and has zero expectation.*

Proof. Fix $t>s>0$ and $B\in\mathcal{B}(\mathbb{R}^d\setminus\{0\})$ such that $\overline{B}\cap\{0\}=\emptyset$, which insures that $\tilde{N}_\omega(t,B)=N_\omega(t,B)-t\nu(B)$ is finite. The last part of the statement is clear. To show the martingale property we see that

$$\begin{aligned}E\Big[\tilde{N}_\omega(t,B)|\mathcal{F}_s\Big]&=E\Big[N_\omega(t,B)-t\nu(B)|\mathcal{F}_s\Big]\\&=E\Big[N_\omega(t,B)-t\nu(B)-N_\omega(s,B)+s\nu(B)+N_\omega(s,B)-s\nu(B)|\mathcal{F}_s\Big]\\&=E\Big[(N_\omega(t,B)-N_\omega(s,B))-(t\nu(B)-s\nu(B))+N_\omega(s,B)-s\nu(B)|\mathcal{F}_s\Big]\\&=E\Big[N_\omega(t,B)-N_\omega(s,B)\Big]-\big(t\nu(B)-s\nu(B)\big)+N_\omega\big(s,B\big)-s\nu(B)\\&=\tilde{N}_\omega\big(s,B\big).\end{aligned}$$

The fourth equality comes from independence and measurability. ∎

We will spend the remainder of this section describing the integration with respect to each of the measures defined above, and the space of functions which are integrable with respect to each measure.

The definition of the Poisson integral is very natural,

$$\int_B f(x)N_\omega(t,dx):=\sum_{x\in B}f(x)N_\omega(t,\{x\}). \tag{2.1}$$

Since for each x, $N_\omega\big(t,\{x\}\big)\neq 0\iff\Delta X_\omega(u)=x$ for some $u\in(0,t]$, we are able to write the integral in the form

$$\int_B f(x)N_\omega(t,dx)=\sum_{0\le s\le t}f\Big(\Delta X_\omega(s)\Big)1_B\Big(\Delta X_\omega(s)\Big). \tag{2.2}$$

Remark 2.1.1. *If we are given that $N_\omega(t,\cdot)$ is a Poisson measure then $G_\omega(t) := \int_B xN_\omega(t,dx)$ is a compound Poisson process and the integral is of the form*

$$\int_B f(x)N_\omega(t,dx) = \sum_{0\le s\le t} f\Big(\Delta G_\omega(s)\Big)1_B\Big(\Delta G_\omega(s)\Big). \tag{2.3}$$

We will now consider this integral in terms of stopping times. For a set $B \in \mathcal{B}(\mathbb{R}^d\backslash\{0\})$, define $T_0^B = 0$ and $T_n^B = \inf\Big\{s > T_{n-1}^B : \Delta X_\omega(s) \in B\Big\}$. Hence $\Big(T_n^B\Big)_{n\in\mathbb{N}}$ are the **arrival times** when X has a jump the size of an element in B. One can see that the arrival times are also stopping times. Using the arrival times, we have an equivalent representation for the integral,

$$\int_B f(x)N_\omega(t,dx) = \sum_{n\in\mathbb{N}} f\bigg(\Delta X_\omega\Big(T_n^B\Big)\bigg)1_{[0,t]}\Big(T_n^B\Big). \tag{2.4}$$

We will now move on to define a class of functions which will be integrable with this measure.

Definition 2.1.3. *Fix $B \in \mathcal{B}\big(\mathbb{R}^d\backslash\{0\}\big)$ and $T > 0$. Take $F\colon [0,T]\times B\times\Omega \to \mathbb{R}$ that satisfies:*

1. *for each $0 \le t \le T$, the mapping $(x,\omega) \to F(t,x,\omega)$ is $\mathcal{B}(B)\otimes\mathcal{F}_t$-measurable;*

2. *for each $x \in B$ and $\omega \in \Omega$, the mapping $t \to F(t,x,\omega)$ is left continuous.*

*Let $\mathcal{P}$ be the smallest σ-algebra with respect to these mappings. We call $\mathcal{P}$ the **predictable σ-algebra** and a $\mathcal{P}$-measurable function is said to be **predictable**.*

Taking $K(t,x,\omega)$ to be a predictable mapping, we further generalize the integral by defining

$$\int_0^T\int_B K\big(t,x,\omega\big)N_\omega(dt,dx) = \sum_{0\le s\le T} K\Big(s,\Delta X_\omega(s),\omega\Big)1_B\Big(\Delta X_\omega(s)\Big). \tag{2.5}$$

Define $\rho\Big((0,t]\times B\Big) = t\nu(B)$ and consider all functions $F\colon [0,T]\times B\times\Omega \to \mathbb{R}$ where F is predictable and $\int_0^T\int_B E\Big[|F(t,x)|^2\Big]\rho(dt,dx) < \infty$. Define $\mathcal{H}_2(T,B)$ to be the space of these functions with the equivalence class set by $\rho\times P$.

Lemma 2.1.2. *([2], page 193) $\mathcal{H}_2(T,B)$ is a Hilbert space.*

We will now work on making the integral with respect to ρ more tangible. We call a function $F(\cdot,\cdot)$ **simple** if for $m, n \in \mathbb{N}$ such that $0 \le t_1 \le \ldots \le t_{m+1} = T$, and $B_1, B_2, \ldots, B_n$ are disjoint Borel subsets of B with $\nu(B_i) < \infty$, we have

$$F(\cdot,\cdot) = \sum_{j=1}^{m}\sum_{k=1}^{n} \tilde{F}_{t_j} 1_{(t_j,t_{j+1}]}(\cdot) 1_{B_k}(\cdot),$$

where $\tilde{F}_{t_j}$ is a bounded $\mathcal{F}_{t_j}$-measurable random variable. Note that $F(\cdot,\cdot)$ is a left-continuous and $\mathcal{B}(B) \otimes \mathcal{F}_t$-measurable and thus predictable. Denote the space of simple functions as $\mathcal{S}(T,B)$.

Proposition 2.1.1. *([2], page 194) $\mathcal{S}(T,B)$ is a dense subset of $\mathcal{H}_2(T,B)$ with respect to the $L^2\Big([0,T) \times B \times \Omega, \rho \times P\Big)$ norm.*

By the proposition above, for any $F \in \mathcal{H}_2(T,B)$ there is a sequence $\{F_n\} \subset \mathcal{S}(T,B)$ such that

$$\int_0^T \int_B F(t,x)\rho(dt,dx) = \lim_{n\to\infty} \int_0^T \int_B F_n(t,x)\rho(dt,dx),$$

with respect to the L^2 norm stated above. We are ready to inject a random jump term into the stochastic replicator dynamics.

2.2 Construction of the Model

Recall that $N(dt,dx)$ is a Poisson measure and $\tilde{N}(dt,dx)$ is a compensated Poisson measure generated by a Lévy process $X = \big(X(t), t \ge 0\big)$ with respect to $\big(\Omega, \mathcal{F}, (\mathcal{F}_t, t \ge 0), P\big)$. To be meticulous, we will assume $\mathcal{F}_0$ contains all of the null sets of $\mathcal{F}$, $\mathcal{F}_t = \mathcal{F}_{t^+} = \bigcap_{u>t} \mathcal{F}_u$, and for every i, W_i is adapted to this filtration.

We adjust the Fudenberg-Harris model [7] by adding a jump term into the payoff function. Hence, we begin with a jump-SDE of the form

$$dr_i(t) = r_i(t-)\left(u_i(\mathbf{r}(\mathbf{t}-))dt + \sigma_i dW_i(t) + \int_{\mathbb{R}} h_i(x)\tilde{N}(dt,dx)\right), \tag{2.6}$$

where $r_i h_i \in \mathcal{H}_2(T,\mathbb{R})$, for $T = \infty$, and $\inf\{h_i(x)\} > -1$. (Note that we are taking the left limit in order to make the coefficients left continuous.) For simplicity we will assume the set B is bounded and bounded below.

Definition 2.2.1. *We call $h_i(x)$ the **jump-affect** of the i^{th} subpopulation.*

Assumption 2.2.1. *For all i:*

a. $\inf_{x\in\mathbb{R}} \left\{h_i(x)\right\} > -1$*;*

b. $h_i(x)$ *is continuously differentiable;*

c. $h_i(x)$ *is bounded.*

Remark 2.2.1. *We are applying the compensated Poisson measure instead of the Poisson measure because this measure disappears in the expectation, and what is left in the expectation is the classic replicator equation, with some stochastic shocks.*

Remark 2.2.2. *Applying Itô's lemma, we can show that* $r_i(t)$ *is of the form* $r_i(t) = \exp\big(Y_i(t)\big)$ *where*

$$dY_i(t) = \left(u_i(\mathbf{r}(t-)) - \frac{\sigma_i^2}{2}\right)dt + \sigma_i dW_i(t) + \int_{\mathbb{R}} \log\left[1+h(x)\right]\tilde{N}(dt,dx) + \int_{\mathbb{R}} \Big(\log\left[1+h(x)\right] - h(x)\Big)\nu(dx)dt.$$

Recalling that $u_i(\mathbf{r}(t)) = \left(A\left(\frac{r_1(t)}{R(t)},\ldots,\frac{r_n(t)}{R(t)}\right)^T\right)_i$, our jump-SDE is further written as

$$dr_i(t) = r_i(t-)\left(\left(A\left(\frac{r_1(t-)}{R(t-)},\ldots,\frac{r_n(t-)}{R(t-)}\right)^T\right)_i dt + \sigma_i dW_i(t) + \int_B h_i(x)\tilde{N}(dt,dx)\right) \ \forall\ i. \tag{2.7}$$

The claim below is a nice consequence of the growth process.

Claim 2.2.1. *Fix* $T > 0$*. Taking our assumptions above for* $r_i(t)$*, we have that*

$$E[r_i(T)] = E\left[\int_0^T r_i(t)u_i(\mathbf{r}(t-))dt\right]$$

Proof. Using an appropriate sequence of simple functions for both the Itô and compensated Poisson integrals we see that these terms will disappear in the expectation. □

Remark 2.2.3. *The intuition behind the jump term is that it captures changes in the payoffs that have come about by sudden impacts in the environment. Consider the example of a species living on a volcanic island and it's strategies to get water. After a volcanic eruption, the payoffs for these strategies change due to the sudden pollution caused* *by the ash that came from eruptions, a blocked route from the rivers of magma, etc. For a particular and pertinent example, the payoff of the strategies for pelicans in the Gulf of Mexico to get food has changed drastically due to the impact of the oil leak. The market crash of 2008 is another great example of a sudden impact that caused changes in the payoffs of the strategies for the traders.*

The next step will be to normalize these processes and apply Itô's lemma. Denote $[\cdot,\cdot]$ as the quadratic variation, $r_i(t)_c$ as the continuous part and $r_i(t)_d$ is the discontinuous part. So for $\mathbf{s}(t) = \left(\frac{r_1(t)}{r_1(t)+\ldots+r_n(t)},\ldots,\frac{r_n(t)}{r_1(t)+\ldots+r_n(t)}\right)$, Itô's lemma gives

$$
\begin{aligned}
ds_i(t) &= \sum_{j\neq i}\frac{-r_i(t-)}{R(t-)^2}dr_j(t)_c + \left(\frac{1}{R(t-)}+\frac{-r_i(t-)}{R(t-)^2}\right)dr_i(t)_c \\
&+\frac{1}{2}\sum_{j\neq i}\frac{2r_i(t-)}{R(t-)^3}d[r_j(t)_c,r_j(t)_c]+\frac{1}{2}\left(\frac{-1}{R(t-)^2}+\frac{-1}{R(t-)^2}+\frac{2r_i(t-)}{R(t-)^3}\right)d[r_i(t)_c,r_i(t)_c] \\
&+\int_{\mathbb{R}}\Bigg(\frac{r_i(t-)+r_i(t-)h_i(x)}{(r_1(t-)+r_1(t-)h_1(x))+(r_2(t-)+r_2(t-)h_2(x))+\ldots+(r_n(t-)+r_n(t-)h_n(x))} \\
&-\frac{r_i(t-)}{r_1(t-)+r_2(t-)+\ldots+r_n(t-)}\Bigg)\tilde{N}(dt,dx) \\
&+\int_{\mathbb{R}}\Bigg(\frac{r_i(t-)+r_i(t-)h_i(x)}{(r_1(t-)+r_1(t-)h_1(x))+(r_2(t-)+r_2(t-)h_2(x))+\ldots+(r_n(t-)+r_n(t-)h_n(x))} \\
&-\frac{r_i(t-)}{r_1(t-)+r_2(t-)+\ldots+r_n(t-)}-\sum_{j\neq i}r_j(t-)h_j(x)\frac{-r_i(t-)}{R(t-)^2} \\
&-r_i(t-)h_i(x)\left(\frac{1}{R(t-)}+\frac{-r_i(t-)}{R(t-)^2}\right)\Bigg)\nu(dx)dt.
\end{aligned}
$$

Substituting $r_i(t) = s_i(t)R(t)$, we get

$$
\begin{aligned}
ds_i(t) &= \sum_{j\neq i}s_i(t-)s_j(t-)\Big[(As(t-))_i-(As(t-))_j)dt+(s_j(t-)\sigma_j^2-s_i(t-)\sigma_i^2)dt \\
&+(\sigma_i dW_i(t)-\sigma_j dW_j(t))\Big] \\
&+\int_{\mathbb{R}}\Bigg(\frac{s_i(t-)+s_i(t-)h_i(x)}{(s_1(t-)+s_1(t-)h_1(x))+(s_2(t-)+s_2(t-)h_2(x))+\ldots+(s_n(t-)+s_n(t-)h_n(x))} \\
&-\frac{s_i(t-)}{s_1(t-)+s_2(t-)+\ldots+s_n(t-)}\Bigg)\tilde{N}(dt,dx) \\
&+\int_{\mathbb{R}}\Bigg(\frac{s_i(t-)+s_i(t-)h_i(x)}{(s_1(t-)+s_1(t-)h_1(x))+(s_2(t-)+s_2(t-)h_2(x))+\ldots+(s_n(t-)+s_n(t-)h_n(x))} \\
&-\frac{s_i(t-)}{s_1(t-)+s_2(t-)+\ldots+s_n(t-)}-\sum_{j\neq i}[s_j(t-)h_i(x)s_i(t-)-s_j(t-)h_j(x)s_i(t-)]\Bigg)\nu(dx)dt.
\end{aligned}
$$

After collecting the terms we arrive at

$$
\begin{aligned}
ds_i(t) = \Bigg[& \sum_{j\neq i} s_i(t-)s_j(t-)((As(t-))_i - (As(t-))_j) + (s_j(t-)\sigma_j^2 - s_i(t-)\sigma_i^2) \\
& + \int_{\mathbb{R}} \Bigg(\frac{s_i(t-) + s_i(t-)h_i(x)}{(s_1(t-) + s_1(t-)h_1(x)) + \ldots + (s_n(t-) + s_n(t-)h_n(x))} \\
& - \frac{s_i(t-)}{s_1(t-) + \ldots + s_n(t-)} - \sum_{j\neq i} [s_j(t-)h_i(x)s_i(t-) - s_j(t-)h_j(x)s_i(t-)] \Bigg) \nu(dx) \Bigg] dt \\
& + \sum_{j\neq i} s_i(t-)s_j(t-)(\sigma_i dW_i(t) - \sigma_j dW_j(t)) \\
& + \int_{\mathbb{R}} \Bigg(\frac{s_i(t-) + s_i(t-)h_i(x)}{(s_1(t-) + s_1(t-)h_1(x)) + \ldots + (s_n(t-) + s_n(t-)h_n(x))} \\
& - \frac{s_i(t-)}{s_1(t-) + \ldots + s_n(t-)} \Bigg) \tilde{N}(dt, dx).
\end{aligned} \tag{2.8}
$$

Note 2.2.1. *In the derivation above, we are missing the term* $\frac{1}{2} \sum_{\substack{j\neq i, k\neq i \\ j\neq k}} \frac{2r_i(t-)}{R(t-)^3} d\big[r_j(t)_c, r_k(t)_c\big]$. *However, since the Wiener processes are pairwise independent,* $d\big[r_j(t)_c, r_k(t)_c\big] = 0$.

We need to be careful above with the domain of the process and make sure that $\mathbf{s}(t)$ is strictly in the simplex, which we define the n-dimensional simplex as Δ_n. If this is not the case, we would need to add a reflection condition on the boundary. However, since r_i has exponential growth (Remark 2.2.3) this should be the case. The claim below advances this fact.

Proposition 2.2.1. *We have that* $P\Big(\mathbf{s}(t) \in \Delta_n \text{ for all } t \geq 0\Big) = 1$.

Proof. We will first show that for $t \geq 0$, we have that $\sum_{i=1}^n s_i(t) = 1$ a.s. Define the map $(s_1(t), \ldots, s_n(t)) \to \sum_{i=1}^n s_i(t) := G(t)$, and set $G(0) = 1$. Itô's lemma gives us that

$$
\begin{aligned}
d(G(t)) = & \sum_{i=1}^n ds_i(t)_c + \int_{\mathbb{R}} \Bigg(\sum_{i=1}^n s_i(t-) + \sum_{i=1}^n \frac{s_i(t-) + s_i(t-)h_i(x)}{(s_1(t-) + s_1(t-)h_1(x)) + \ldots + (s_n(t-) + s_n(t-)h_n(x))} \\
& - \sum_{i=1}^n \frac{s_i(t-)}{s_1(t-) + \ldots + s_n(t-)} - \sum_{i=1}^n s_i(t-) \Bigg) \tilde{N}(dt, dx) \\
= & \sum_{i=1}^n \Bigg[\sum_{j\neq i} s_i(t-)s_j(t-)((As(t-))_i - (As(t-))_j) + (s_j(t-)\sigma_j^2 - s_i(t-)\sigma_i^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}}\Bigg(\frac{s_i(t-)+s_i(t-)h_i(x)}{(s_1(t-)+s_1(t-)h_1(x))+\ldots+(s_n(t-)+s_n(t-)h_n(x))}\\
&-\frac{s_i(t-)}{s_1(t-)+\ldots+s_n(t-)}-\sum_{j\neq i}[s_j(t-)h_i(x)s_i(t-)-s_j(t-)h_j(x)s_i(t-)]\Bigg)\nu(dx)\Bigg]\,dt\\
&+\sum_{i=1}^{n}\sum_{j\neq i}s_i(t-)s_j(t-)(\sigma_i dW_i(t)-\sigma_j dW_j(t))\\
&=\left[\sum_{i=1}^{n}\sum_{j\neq i}s_i(t-)s_j(t-)As(t-))_i-\sum_{i=1}^{n}\sum_{j\neq i}s_i(t-)s_j(t-)(As(t-))_j\right]dt\\
&+\left[\sum_{i=1}^{n}\sum_{j\neq i}s_i(t-)s_j(t-)s_j(t-)\sigma_j^2-\sum_{i=1}^{n}\sum_{j\neq i}s_i(t-)s_j(t-)s_i(t-)\sigma_i^2\right]dt\\
&+\int_{\mathbb{R}}\Bigg(\sum_{i=1}^{n}\frac{s_i(t-)+s_i(t-)h_i(x)}{(s_1(t-)+s_1(t-)h_1(x))+\ldots+(s_n(t-)+s_n(t-)h_n(x))}\\
&-\sum_{i=1}^{n}\frac{s_i(t-)}{s_1(t-)+\ldots+s_n(t-)}-\sum_{i=1}^{n}\sum_{j\neq i}s_j(t-)h_i(x)s_i(t-)+\sum_{1=i}^{n}\sum_{j\neq i}s_j(t-)h_j(x)s_i(t-)\Bigg)\nu(dx)dt\\
&+\sum_{i=1}^{n}\sum_{j\neq i}s_i(t-)s_j(t-)\sigma_i dW_i(t)-\sum_{i=1}^{n}\sum_{j\neq i}s_i(t-)s_j(t-)\sigma_j dW_j(t)\\
&=0\quad a.s.
\end{aligned}
$$

Recall that since the Brownian motions are pairwise independent, by the nature of the mapping, the second term of the continuous part of the process is just zero. Furthermore, the first and second sums in the integral both add up to 1 and hence they cancel. Therefore, by the initial condition we are done.

Finally, we show that the process does not hit or jump over the boundary in finite time. Define $\Psi:\Delta_n\to\mathbb{R}^{n-1}$ by $\Psi(\mathbf{y})=\Big(\log(y_1/y_n),\log(y_2/y_n),\ldots,\log(y_{n-1}/y_n)\Big)^T$. Furthermore, take τ to be the first time the process leaves the simplex (i.e., such that $s_i(t)\leq 0$ for some i) and take $Z(t):=\Psi(\mathbf{s}(t))$ for $t<\tau$. Itô's lemma (and applying the equality $s_1(t)+\ldots+s_n(t)=1$) yields

$$
\begin{aligned}
dZ_i(t)&=\sum_j s_j(t)_c\frac{\partial\Psi_i}{\partial y_j}(\mathbf{s}(t))+\frac{1}{2}\sum_{j,k}s_j(t)_c s_k(t)_c\frac{\partial^2\Psi_i}{\partial y_j\partial y_k}(\mathbf{s}(t))\\
&-\int_{\mathbb{R}}\sum_j H_i\big(s(t),h(x)\big)\frac{\partial\Psi_i}{\partial y_j}(\mathbf{s}(t))\nu(dx)dt+\int_{\mathbb{R}}\Big(\Psi_i\Big(H\big(\mathbf{s}(t),\mathbf{h(x)}\big)+\mathbf{s}(t)\Big)-\Psi_i(\mathbf{s}(t))\Big)\tilde{N}(dx,dt)\\
&+\int_{\mathbb{R}}\Big(\Psi_i\Big(H\big(\mathbf{s}(t),\mathbf{h(x)}\big)+\mathbf{s}(t)\Big)-\Psi_i(\mathbf{s}(t))\Big)\nu(dx)dt,
\end{aligned}
$$

where $H\big(\mathbf{y},\mathbf{h}(\mathbf{x})\big) = \dfrac{1}{(\mathbf{y}+\mathbf{y}\mathbf{h}(\mathbf{x})^T)\cdot\mathbf{1}}(\mathbf{y}+\mathbf{y}\mathbf{h}(\mathbf{x})^T) - \dfrac{1}{\mathbf{y}\cdot\mathbf{1}}\mathbf{y}$. Thus

$$
\begin{aligned}
dZ_i(t) = &\Bigg[\big(As(t-)\big)_i - \sum_j s_j(t-)\big(As(t-)\big)_j + \sum_j s_j(t-)\sigma_j^2 - s_i(t-)\sigma_i^2 \\
&+ \int_{\mathbb{R}} \left(\frac{1+h_i(x)}{1+\sum_j s_j(t-)h_j(x)} - 1 + \sum_j s_j(t-)h_j(x) - h_i(x) \right) \nu(dx) \Bigg] dt \\
&+ \left(\sigma_i dW_i(t) - \sum_j s_j(t-)\sigma_j dW_j(t) \right) \\
&- \Bigg[\big(As(t-)\big)_n - \sum_j s_j(t-)\big(As(t-)\big)_j + \sum_j s_j(t-)\sigma_j^2 - s_n(t-)\sigma_i^2 \\
&+ \int_{\mathbb{R}} \left(\frac{1+h_n(x)}{1+\sum_j s_j(t-)h_j(x)} - 1 + \sum_j s_j(t-)h_j(x) - h_n(x) \right) \nu(dx) \Bigg] dt \\
&- \left(\sigma_n dW_n(t) - \sum_j s_j(t-)\sigma_j dW_j(t) \right) \\
&- \frac{1}{2}\left(\sigma_i dW_i(t) - \sum_j s_j(t-)\sigma_j dW_j(t) \right)^2 + \frac{1}{2}\left(\sigma_n dW_n(t) - \sum_j s_j(t-)\sigma_j dW_j(t) \right)^2 \\
&- \int_{\mathbb{R}} \left[\left(\frac{1+h_i(x)}{1+\sum_j s_j(t-)h_j(x)} - 1 \right) - \left(\frac{1+h_n(x)}{1+\sum_j s_j(t-)h_j(x)} - 1 \right) \right] \nu(dx)dt \\
&+ \int_{\mathbb{R}} \left(\log\left(\frac{s_i(t-) + \frac{s_i(t-)+s_i(t-)h_i(x)}{1+\sum_j s_j(t-)h_j(x)} - s_i(t-)}{s_n(t-) + \frac{s_n(t-)+s_n(t-)h_n(x)}{1+\sum_j s_j(t-)h_j(x)} - s_n(t-)} \right) - \log\big(s_i(t-)/s_n(t-)\big) \right) \tilde{N}(dx,dt) \\
&+ \int_{\mathbb{R}} \left(\log\left(\frac{\frac{s_i(t-)+s_i(t-)h_i(x)}{1+\sum_j s_j(t-)h_j(x)}}{\frac{s_n(t-)+s_n(t-)h_n(x)}{1+\sum_j s_j(t-)h_j(x)}} \right) - \log\big(s_i(t-)/s_n(t-)\big) \right) \nu(dx)dt.
\end{aligned}
$$

Rearranging and simplifying, we arrive at

$$
\begin{aligned}
dZ_i(t) = &\left[\Big(\mathbf{A}\mathbf{s}(t-)\Big)_i - \Big(\mathbf{A}\mathbf{s}(t-)\Big)_n + \frac{1}{2}(\sigma_n^2 - \sigma_i^2) + \int_{\mathbb{R}} \big(h_n(x) - h_i(x)\big)\nu(dx) \right] dt \\
&+ \sigma_i dW_i(t) - \sigma_n dW_n(t) + \int_{\mathbb{R}} \left(\log\left(\frac{s_i(t-)+s_i(t-)h_i(x)}{s_n(t-)+s_n(t-)h_n(x)} \right) - Z_i(t-) \right) \tilde{N}(dx,dt) \\
&+ \int_{\mathbb{R}} \left(\log\left(\frac{s_i(t-)+s_i(t-)h_i(x)}{s_n(t-)+s_n(t-)h_n(x)} \right) - Z_i(t-) \right) \nu(dx)dt \\
= &\left[(\mathbf{e}_i - \mathbf{e}_n)^T A \Psi^{-1}(Z(t-)) + \frac{1}{2}(\sigma_n^2 - \sigma_i^2) + \int_{\mathbb{R}} \left(h_n(x) - h_i(x) + \log\left(\frac{1+h_i(x)}{1+h_n(x)} \right) \right) \nu(dx) \right] dt \\
&+ \sigma_i dW_i(t) - \sigma_n dW_n(t) + \int_{\mathbb{R}} \log\left(\frac{1+h_i(x)}{1+h_n(x)} \right) \tilde{N}(dx,dt),
\end{aligned}
\tag{2.9}
$$

where

$$s_j(t) = \frac{s_j(t)}{s_1(t)+s_2(t)+\ldots+s_n(t)} = \frac{s_j(t)/s_n(t)}{s_1(t)/s_n(t)+s_2(t)/s_n(t)+\ldots+1} = \frac{e^{Z_j(t)}}{e^{Z_1(t)}+e^{Z_2(t)}+\ldots+1}$$

and

$$\Psi^{-1}(\mathbf{y}) = \frac{1}{e^{y_1}+e^{y_2}+\ldots+1}(e^{y_1}, e^{y_2}, \ldots, 1)^T.$$

Define L as the infinitesimal generator for $\mathbf{Z}(t)$. Applying Theorem 2.1 in Meyn and Tweedie [17], we need to show that there exists a $\varphi \in C^2$ such that $\lim\limits_{|\mathbf{y}|\to\infty} \varphi\big(\mathbf{y}\big) = \infty$ and there is a positive constant λ where $L\varphi \leq \lambda\varphi$ so that $P(\tau = \infty) = 1$. Since our coefficients are time homogeneous, for $t \geq 0$, our generator has the form

$$\begin{aligned}
L\varphi(\cdot) = {} & \sum_{i=1}^{n-1} a_i(\cdot)\frac{\partial\varphi}{\partial y_i}(\cdot) + \frac{1}{2}\sum_{j,k=1}^{n-1} b_{jk}\frac{\partial\varphi}{\partial y_i}(\cdot) \\
& + \int_{\mathbb{R}} \left[\varphi\Bigg(\cdot\ + \Bigg(\log\left(\frac{1+h_1(x)}{1+h_n(x)}\right), \ldots, \log\left(\frac{1+h_{n-1}(x)}{1+h_n(x)}\right)\Bigg)\Bigg) - \varphi(\cdot)\right]\nu(dx),
\end{aligned}$$

where

$$a_i(y) = \big(\mathbf{e}_i - \mathbf{e}_n\big)^T A\Psi^{-1}(y) + \frac{1}{2}\big(\sigma_n^2 - \sigma_i^2\big) + \int_{\mathbb{R}} \left(h_n(x) - h_i(x)\right)\nu(dx)$$

and

$$b_{jk} = \begin{cases} \sigma_j^2 + \sigma_n^2 & \text{if } j = k \\ \sigma_n^2 & \text{if } j \neq k \end{cases}.$$

Note that $a_i(y)$ is bounded by a constant. Now taking $\varphi(\mathbf{y}) = 1 + |\mathbf{y}|^2$, it is clear that $L\varphi(\mathbf{y})$ is a linear term, and thus there exists a λ where the inequality holds. Thus by Theorem 2.1 in Meyn and Tweedie [17], $P\big(\tau = \infty\big) = 1$, and we are done. □

We will first consider a two subpopulation environment. Recalling that $s_2(t) = 1 - s_1(t)$, our focus just needs to be on analyzing $s_1(t)$. Similarly to Fudenberg and Harris [7], we have the jump-SDE

$$\begin{aligned}
ds_1(t) = {} & s_1(t-)\big(1-s_1(t-)\big)\left[\big(a_{11}-a_{21}\big)s_1(t-) + \big(a_{12}-a_{22}\big)\big(1-s_1(t-)\big) + \sigma_2^2\big(1-s_1(t-)\big) - \sigma_1^2 s_1(t-)\big)\right]dt \\
& + \sigma s_1(t-)\big(1-s_1(t-)\big)dW(t) \\
& + \int_{\mathbb{R}} \left(\frac{s_1(t-)+s_1(t-)h_1(x)}{\left(s_1(t-)+s_1(t-)h_1(x)\right)+\big(s_2(t-)+s_2(t-)h_2(x)\big)} - \frac{s_1(t-)}{s_1(t-)+s_2(t-)}\right)\tilde{N}(dt,dx) \\
& + \int_{\mathbb{R}} \left(\frac{s_1(t-)+s_1(t-)h_1(x)}{\left(s_1(t-)+s_1(t-)h_1(x)\right)+\big(s_2(t-)+s_2(t-)h_2(x)\big)} - \frac{s_1(t-)}{s_1(t-)+s_2(t-)}\right. \\
& \left. - \frac{s_1(t-)h_1(x)s_2(t-) - s_2(t-)h_2(x)s_1(t-)}{\big(s_1(t-)+s_2(t-)\big)^2}\right)\nu(dx)dt,
\end{aligned} \tag{2.10}$$

recalling that $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ and $W(t) = \big(\sigma_1 W_1(t) - \sigma_2 W_2(t)\big)/\sigma$. Simplifying the integrals we have the alternate expression

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \frac{s_1(t-)\big(1 - s_1(t-)\big)\big[h_1(x) - h_2(x)\big]}{s_1(t-)\big[h_1(x) - h_2(x)\big] + 1 + h_2(x)} \tilde{N}(dt, dx) \\ &+ \int_0^T \int_{\mathbb{R}} \Bigg(\frac{s_1(t-)\big(1 - s_1(t-)\big)\big[h_1(x) - h_2(x)\big]}{s_1(t-)\big[h_1(x) - h_2(x)\big] + 1 + h_2(x)} \\ &+ s_1(t-)\big(1 - s_1(t-)\big)\big[h_2(x) - h_1(x)\big] \Bigg) \nu(dx)dt.\end{aligned} \tag{2.11}$$

Notice that the $\nu(dx)dt$ integral in equation (2.10) can be absorbed into the Lebesgue integral in our model. For simplicity of the notation, we will use $s(t)$ instead of $s_1(t)$. Therefore our SDE is of the form

$$ds(t) = \alpha\big(s(t-)\big)dt + \beta\big(s(t-)\big)dW(t) + \int_{\mathbb{R}} K_1\big(s(t-), x\big)\tilde{N}(dt, dx),$$

where

$$\alpha(y) = y(1-y)\left[-a_{22} + a_{12} + \sigma_2^2 + \int_{\mathbb{R}} K_2(y,x)\nu(dx) + (a_{11} - a_{21} - \sigma_1^2 + a_{22} - a_{12} - \sigma_2^2)y\right]$$

$$\beta(y) = \sigma y\big(1-y\big),$$

$$K_1(y,x) = \frac{y\big(1-y\big)\big[h_1(x) - h_2(x)\big]}{y\big[h_1(x) - h_2(x)\big] + 1 + h_2(x)}$$

and

$$K_2(y,x) = \frac{y\big(1-y\big)\big[h_1(x) - h_2(x)\big]}{y\big[h_1(x) - h_2(x)\big] + 1 + h_2(x)} + \big[h_2(x) - h_1(x)\big]y\big(1-y\big).$$

Chapter 3

Analysis of the Model with the Stochastic Lyapunov Method

3.1 An overview of the Stochastic Lyapunov Method

We will prove the applicable lemmas for a theorem given in Kushner in order to comprehend the method. Since this concept might be a bit foreign, we will first discuss the ideas behind the proofs in order to get an understanding of how we will proceed. Do note though that this is similar to the classical Lyapunov method. Take x_t to be a strong right-continuous Markov process, $\mathcal{A}$ the infinitesimal generator $\left(\mathcal{A}(f)(x) = \lim_{t \searrow 0} \frac{\mathbb{E}_x[f\big(x(t)\big)] - f(x)}{t}\right)$ with respect to the supremum norm $||\cdot||_\infty$, $V(x)$ a positive function and τ a stopping (Markov) time with $\mathbb{E}_x[\tau] < \infty$. If we have that $\mathcal{A}V(x) := -k(x) \leq 0$, then by Dynkin's formula,

$$V(x) - \mathbb{E}_x[V(x(\tau))] = \mathbb{E}_x\left[\int_0^\tau k\big(x(s)\big)ds\right] \geq 0.$$

A result in Dynkin ([5], Volume 2) tells us that if $V(x) \geq \mathbb{E}_x[V\big(x(\tau)\big)]$ and $\mathbb{E}_x[V\big(x(t\wedge\tau)\big)] \to V(x)$ as $t \to 0$ then $V\big(x(t\wedge\tau)\big)$ is a supermartingale. Taking $\tau \to \infty$ a.s., we have that the processes $V\big(x(t)\big) \to c(\omega) \geq 0$ as $t \to \infty$, for some random variable $c(\omega)$. Moreover, as $t \to \infty$, $x(t) \to \big\{x : k(x) = 0\big\}$. We will now move on to the specifics of this method.

The following is a list of assumptions that we will use in this section:

(3A) $V(x)$ is nonnegative and continuous in the open set $Q_m := \{x : V(x) < m\}$.

(3B) $x(t)$ is a strong right-continuous Markov process and $\tau_m := \inf\{t : x(t) \notin Q_m\}$. Take $\tau_m(\omega) = \infty$ a.s., if $x_\omega(t) \in Q_m$ for all $t < \infty$.

(3C) V is in the domain of $\mathcal{A}_m$, where $\mathcal{A}_m$ is the infinitesimal generator over Q_m.

(3D) For any $x \in Q_m$ and $\epsilon > 0$, $P_x\left(\sup_{s\leq t} ||x(s) - x|| > \epsilon\right) \to 0$ as $t \to 0$.

Furthermore, for sets M and Q, and $\epsilon > 0$, we define $N_{Q,\epsilon}(M) := \Big\{x \in Q : ||x - y|| \leq \epsilon \text{ for some } y \in M\Big\}$. Lastly, denote B_m by the set of ω such that $x(t) \in Q_m$ for all $t < \infty$.

Lemma 3.1.1. *([14], page 37) Assume 3A, 3B, 3C, and that $\mathcal{A}_m V(x) \leq 0$, and take $x_0 \in Q_m$ as the initial condition. Then $V\big(x(t \wedge \tau_m)\big)$ is a nonnegative supermartingale of the process $\Big(x(t \wedge \tau_m)\Big)_{t \in \mathbb{R}_+}$, and for $\lambda \leq m$,*

$$P_{x_0}\left(\sup_{0<t<\infty} V\big(x(t \wedge \tau_m)\big) \geq \lambda\right) \leq \frac{V(x_0)}{\lambda}.$$

Furthermore, there is a random variable $0 \leq c(\omega) < m$, where, as $t \to \infty$, $V\big(x(t)\big) \to c(\omega)$ a.s. on B_m. Moreover,

$$P_{x_0}(B_m) \geq 1 - \frac{V(x_0)}{m}.$$

Proof. Dynkin's formula gives us

$$\mathbb{E}_{x_0}\Big[V\big(x(t \wedge \tau_m)\big)\Big] - V(x_0) = \mathbb{E}_{x_0}\left[\int_0^{t \wedge \tau_m} \mathcal{A}V\big(x(s)\big)ds\right] \leq 0,$$

which implies that $\mathbb{E}_{x_0}\Big[V\big(x(t \wedge \tau_m)\big)\Big] \leq V(x_0)$. Furthermore, since $V(x)$ is in the domain of $\mathcal{A}_m$, $V\big(x(t \wedge \tau_m)\big) \to V(x_0)$ as $t \to 0$ a.s., and thus $V\big(x(t \wedge \tau_m)\big)$ is a supermartingale by Dynkin ([8], Theorem 12.6).

The supermartingale inequality yields $P_{x_0}\left(\sup_{0<t<\infty} V\big(x(t \wedge \tau_m)\big) \geq \lambda\right) \leq \frac{V(x_0)}{\lambda}$. For the convergence to $c(\omega)$, we first note that $\left(\mathbb{E}\Big[V\big(x(t \wedge \tau_m)\big)\Big]\right)_{t \in \mathbb{R}_+}$ is positive and decreasing. Hence $\Big(-V\big(x(t \wedge \tau_m)\big)\Big)_{t \in \mathbb{R}_+}$ is a submartingale with $\mathbb{E}\Big[\big|-V\big(x(\tau_m)\big)\big|\Big] < m$, and therefore Doob's submartingale convergence theorem tells us we have convergence in B_m. The last inequality is now clear. □

Lemma 3.1.2. *([14], page 38) Assume the first part of 3B, $V(x)$ nonnegative in the open region Q, $V(x)$ is in the domain of $\mathcal{A}_Q$, τ to be the first exit time out of the open region $P \subset Q$ and finally $\mathcal{A}_Q V(x) \leq -c$ for some positive c in P. Then $\mathbb{E}_{x_0}[\tau] \leq V(x_0)/c$.*

Proof. Applying Dynkin's formula and using the fact that $V(x_{t \wedge \tau})$ is nonnegative, we have

$$\begin{aligned} V(x_0) > V(x_0) - \mathbb{E}_{x_0}\Big[V\big(x(t \wedge \tau)\big)\Big] &= -\,\mathbb{E}_{x_0}\left[\int_0^{t \wedge \tau} \mathcal{A}_Q V\big(x(s)\big)ds\right] \\ &\geq c\mathbb{E}_{x_0}\left[\int_0^{t \wedge \tau} 1_P(x(s))ds\right] \\ &= c\mathbb{E}_{x_0}[t \wedge \tau]. \end{aligned}$$

□

Theorem 3.1.1. *([14], page 39) Assume 3A through 3D, $\mathcal{A}V(x) = -k(x) \leq 0$ and Q_m is bounded. Take x_0 to be the initial condition and define $P_m = Q_m \cap \{x : k(x) = 0\}$. For some $d_0 > 0$, we have for each $0 < d < d_0$, there* *exists an ϵ_d such that $k(x) \geq d$ on $Q_m \backslash N_{Q_m,\epsilon_d}(P_m)$, i.e., we will assume that $k(x)$ is uniformly continuous on P_m and $k(x) > 0$ for some $x \in Q_m$. Then with probability no less than $1 - V(x_0)/m$, $x_t \to P_m$.*

Define $P = \bigcup_{m=1}^{\infty} P_m$ and $Q = \bigcup_{m=1}^{\infty} Q_m$ and assume that the hypotheses above hold for all m. If for each $0 < d < d_0$, there is an $\epsilon_d > 0$ such that $k(x) \geq d$ on $Q \backslash N_{Q,\epsilon_d}(P)$ then $x(t) \to P$ a.s.

Proof. Fix d_1, d_2 such that $d_0 > d_1 > d_2 > 0$ and let ϵ_i correspond to d_i. Without loss of generality, we will suppose that $N_{Q_m,\epsilon_2}(P_m) \subset N_{Q_m,\epsilon_1}(P_m)$. Define

$$T_{x_0}(t, \epsilon_i) = \int_{t \wedge \tau_m}^{\tau_m} 1_{Q_m \backslash N_{Q_m,\epsilon_i}(P_m)}\big(x_{x_0}(s)\big) ds,$$

with the convention $T_{x_0}(t, \epsilon_i) = 0$, if $t > \tau_m$. Hence $T_{x_0}(t, \epsilon_i)$ is the total time spent in $Q_m \backslash N_{Q_m,\epsilon_i}(P_m)$ after some time t and before the first exit time from Q_m, (or possibly $t = \infty$). By Lemma 3.1,

$$P_{x_0}\Big(x(t) \text{ leaves } Q_m \text{ at least once before } t = \infty\Big) = 1 - P_{x_0}(B_m) \leq \frac{V(x_0)}{m}.$$

By Lemma 3.2, with probability one, $T_{x_0}(t, \epsilon_i) < \infty$ and therefore, as $t \to \infty$, $T_{x_0}(t, \epsilon_i) \to 0$ a.s.

We now have two possiblities:

(a) there is a random variable $\tau(\epsilon_1)$ that is finite with probability one such that $x(t) \in N_{Q_m,\epsilon_1}(P_m)$ with probability one, relative to B_m, for all $t > \tau(\epsilon_1)$;

(b) for $\omega \in B_m$, $x(t)$ moves from $N_{Q_m,\epsilon_2}(P_m)$ to $Q_m \backslash N_{Q_m,\epsilon_1}(P_m)$ and back to $N_{Q_m,\epsilon_2}(P_m)$ infinitely often in any time interval $[t, \infty)$.

We will show that (b) has a zero probability. Fix $\delta_1 > 0$ and choose $h > 0$ so that

$$\sup_{x \in Q_m \backslash N_{Q_m,\epsilon_2}(P_m)} P_x\left(\sup_{h \geq s \geq 0} ||x(s) - x|| \geq \epsilon_1 - \epsilon_2\right) < \delta_1,$$

which we know exists by stochastic continuity and the compactness of $\overline{Q}_m$. Moreover, for each $\delta_2 > 0$ and initial condition x_0, there is a $t < \infty$ so that

$$P_{x_0}\Big(T_{x_0}(t, \epsilon_2) > h\Big) < \delta_2.$$

However, both of these are contradictory to (b). Therefore, for $x_0 \in Q_m$,

$$P_{x_0}\Big(x(s) \in Q_m \backslash N_{Q_m,\epsilon_1}(P_m) \text{ for some } s \in (t, \infty)\Big) \to 0 \text{ as } t \to \infty.$$

Since the ϵ_i's are arbitrary, we are done. □

3.2 Stochastic Lyapunov Analysis of the Fudenberg and Harris Model

Recall that the diffusion is strictly in the unit interval. For this model, we will take $\mathcal{A}$ under the $||\cdot||_\infty$ norm. So for $f \in D(\mathcal{A})$ (the domain of $\mathcal{A}$) and $y \in (0,1)$ equation (1.4) gives

$$\begin{aligned}\mathcal{A}(f)(y) &= \lim_{t\searrow 0} \frac{\mathbb{E}[f(s(t)+y)] - f(y)}{t} \\ &= y(1-y)[(a_{12}+\sigma_2^2 - a_{22}) + \{(a_{11}-a_{21}-\sigma_1^2) + (a_{22}-a_{12}-\sigma_2^2)\}y]f'(y) \\ &+ \frac{1}{2}\sigma^2 y^2(1-y)^2 f''(y).\end{aligned} \tag{3.1}$$

In order to make $D(\mathcal{A})$ more tangible, we will show that $C_b^2([0,1]) \subset D(\mathcal{A})$. Since we are working in an interval, the issue of boundary conditions has to be addressed. In doing so we need to consider that the end points 0 and 1 are stationary points, i.e., they are absorbing. Moreover, they are the only stationary points. However, as the claim below will show, the process will not hit the boundary in finite time. For simplicity we will set $a = a_{11} - a_{21} - \sigma_1^2$, $b = a_{22} - a_{12} + a_{21} - \sigma_2^2$ for the remainder of the paper .

Claim 3.2.1. *The process $s(t)$ does not hit the boundary in finite time.*

Proof. This will be shown by Feller's criteria ([13], pages 342-348). For an arbitrary $c \in (0,1)$, define

$$\begin{aligned} p(x) &= \int_c^x \exp\left\{-2\int_c^\eta \frac{\xi(1-\xi)(-b+(a+b)\xi)}{\xi^2(1-\xi)^2\sigma^2}d\xi\right\}d\eta \\ &= \int_c^x \exp\left\{-2\int_c^\eta \frac{(-b+(a+b)\xi)}{\xi(1-\xi)\sigma^2}d\xi\right\}d\eta \\ &= \int_c^x \exp\left\{-\frac{2}{\sigma^2}\int_c^\eta \frac{(-b+(a+b)\xi)}{\xi(1-\xi)}d\xi\right\}d\eta \\ &= \int_c^x \exp\left\{-\frac{2}{\sigma^2}\int_c^\eta \left(\frac{-b}{\xi} + \frac{a}{(1-\xi)}\right)d\xi\right\}d\eta \\ &= \int_c^x \left((\eta/c)^{2b/\sigma^2}\left([1-\eta]/[1-c]\right)^{2a/\sigma^2}\right)d\eta \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} v(x) &= \int_c^x p'(y)\int_c^y \frac{2dz}{p'(z)z^2(1-z)^2\sigma^2}dy \\ &= k\int_c^x (y)^{2b/\sigma^2}(1-y)^{2a/\sigma^2}\int_c^y (z)^{-2b/\sigma^2-2}(1-z)^{-2a/\sigma^2-2}dzdy. \end{aligned}$$

Taking $c = s(0)$, we need to determine that $v(1+) = v(0-) = \infty$. First we will show that $v(1+) = \infty$. Fix an $\epsilon > 0$ to be very small and define

$$m_0 = \min\left\{(y)^{2b/\sigma^2} : y \in [c,1]\right\},$$

and

$$m_1 = \min\Big\{(y)^{-2b/\sigma^2-2} : y \in [c,1]\Big\}.$$

Since these functions are strictly positive in this interval, $m_0, m_1 > 0$.

If $2a/\sigma^2 < -1$ then for an arbitrary d, where $c < d < 1-\epsilon$, we have

$$\begin{aligned}
v(1-\epsilon) &\geq k \int_d^{1-\epsilon} (y)^{2b/\sigma^2}(1-y)^{2a/\sigma^2} \int_c^y (z)^{-2b/\sigma^2-2}(1-z)^{-2a/\sigma^2-2} dz dy \\
&\geq k \int_d^{1-\epsilon} (y)^{2b/\sigma^2}(1-y)^{2a/\sigma^2} \int_c^d (z)^{-2b/\sigma^2-2}(1-z)^{-2a/\sigma^2-2} dz dy \\
&\geq m_0 k_d k \left(\frac{-1}{2a/\sigma^2+1}(\epsilon)^{2a/\sigma^2+1} + \frac{1}{2a/\sigma^2+1}(1-d)^{2a/\sigma^2+1} \right) \to \infty \text{ as } \epsilon \to 0,
\end{aligned}$$

where $k_d := \displaystyle\int_c^d (z)^{-2b/\sigma^2-2}(1-z)^{-2a/\sigma^2-2} dz$.

Now suppose that $2a/\sigma^2 > -1$. Then

$$\begin{aligned}
v(1-\epsilon) &\geq m_0 m_1 k \int_c^{1-\epsilon} (1-y)^{2a/\sigma^2} \int_c^y (1-z)^{-2a/\sigma^2-2} dz dy \\
&= \frac{m_0 m_1 k}{2a/\sigma^2+1} \int_c^{1-\epsilon} [(1-y)^{-1} - (1-y)^{2a/\sigma^2}(1-c)^{-2a/\sigma^2-1}] dy \\
&= \frac{m_0 m_1 k}{2a/\sigma^2+1} \left(-\log(\epsilon) + \frac{(1-c)^{-2a/\sigma^2-1}}{2a/\sigma^2+1}(\epsilon)^{2a/\sigma^2+1} + \log(1-c) - \frac{1}{2a/\sigma^2+1} \right) \to \infty \text{ as } \epsilon \to 0.
\end{aligned}$$

The case of $v(0-) = \infty$ is shown with the same method. Done. $\square$

Define $\hat{\alpha}(x) = x(1-x)(-b+\{a+b\}x)$, $\hat{\beta}(x) = x(1-x)\sigma$ and $\mathtt{m}_\infty$ as the killing time (for our model $P(\mathtt{m}_\infty = \infty) = 1$). Note that $p(x)$ is our scaling function and $m(dx) := \dfrac{2dx}{p'(x)\hat{\beta}(x)^2}$ is our speed measure ([13], pages 342-348). Recalling that 0 and 1 are absorbing, Itô and McKean ([12], pages 105-108), give the conditions

$$\mathcal{A}(f)(0) = -\frac{f(0)}{\mathbb{E}_0[\mathtt{m}_\infty]} = 0$$

and

$$\mathcal{A}(f)(1) = -\frac{f(1)}{\mathbb{E}_1[\mathtt{m}_\infty]} = 0,$$

for f in the domain of $\mathcal{A}$. Notice, for appropriate functions, we are guaranteed this by our infinitesimal generator. Furthermore, Itô and McKean ([12], pages 105-108) tell us we need

$$\mathcal{A}(f)(x)m(dx) = f^+(dx),$$

where $f^+(dx)$ is the Borel measure generated by $f^+(m,n] = f^+(n) - f^+(m)$, and

$$f^+(c) := \lim_{d \searrow c} \frac{f(d) - f(c)}{p(d) - p(c)} = \lim_{d \searrow c} \frac{\frac{f(d)-f(c)}{d-c}}{\frac{p(d)-p(c)}{d-c}} = \frac{f'(c)}{p'(c)}.$$

Hence $f^+(dx) = \left(\frac{f'(c)}{p'(c)}\right)' dx$. Also

$$\begin{aligned}
\mathcal{A}(f)(x)m(dx) &= \left(\frac{2}{p'(x)\hat{\beta}(x)^2}\hat{\alpha}(x)f'(x) + \frac{2}{p'(x)\hat{\beta}(x)^2}\frac{1}{2}\hat{\beta}(x)^2 f''(x)\right) dx \\
&= \left(\left(\frac{2\hat{\alpha}(x)}{\hat{\beta}(x)^2}\frac{1}{p'(x)}\right) \cdot f'(x) + f''(x) \cdot \frac{1}{p'(x)}\right) dx \\
&= \left(\frac{f'(c)}{p'(c)}\right)' dx,
\end{aligned}$$

and therefore we have our equality.

Clearly we need f to be twice continuously differentiable. Moreover if f is bounded on the interval this would forgo any possible complications. Therefore $C_b^2([0,1]) \subset D(\mathcal{A})$.

We will now move on to determining the conditions needed for the stochastic Lyapunov functions. Quite clearly the sign of $y(1-y)[-b+\{a+b\}y]$ is dependent on $-b+\{a+b\}y := \zeta(y)$, thus we will dissect this function. We will determine what values of y will make the sign of $\zeta(y)$ negative/positive, and in what intervals. Our focus will be on $-b < \{-a-b\}y$ and what values of y makes this the inequality true. Before we proceed, we will make a couple of observations: $-b < 0 \implies \sigma_2^2 < a_{22} - a_{12}$; and $-b > 0 \implies \sigma_2^2 > a_{22} - a_{12}$.

The analysis is broken down into two cases:

(I) We first assume $-a-b > 0$, i.e.,

$$\frac{-b}{-a-b} < y \implies \frac{b}{a+b} < 1 \implies 0 > a \implies a_{11} - a_{21} < \sigma_1^2.$$

   i. Hence if $-b < 0$ then we would have that $\zeta(y)$ is always negative, which we have when $\sigma_2^2 < a_{22} - a_{12}$ and $a_{11} - a_{21} < \sigma_1^2$.

   ii. If $-b > 0$ then $\zeta(y)$ is postive for $y \in (0, b/(a+b))$ and negative for $y \in (b/(a+b), 1)$ and we have have this when $\sigma_2^2 > a_{22} - a_{12}$ and $a_{11} - a_{21} < \sigma_1^2$.

(II) Next we have $-a-b < 0$, i.e., $\dfrac{b}{a+b} > y$.

   i. If $-b > 0$ then we would have that $\zeta(y)$ is never negative, and this is the case when we have the inequalities $\sigma_2^2 > a_{22} - a_{12}$ and $a_{11} - a_{21} > \sigma_1^2$.

ii. Lastly, if $-b < 0$ then we have two different scenarios, $\frac{b}{a+b} < 1$ and $\frac{b}{a+b} > 1$. If $\frac{b}{a+b} < 1$ then

$$\frac{b}{a+b} < 1 \implies 0 < a \implies a_{11} - a_{21} > \sigma_1^2.$$

We have $\zeta(y)$ is negative for $y \in (0, b/(a+b))$ and positive for $y \in (b/(a+b), 1)$ and this applies when $\sigma_2^2 < a_{22} - a_{12}$ and $a_{11} - a_{21} > \sigma_1^2$. Finally if $\frac{b}{a+b} > 1$ then we would have $\zeta(y)$ is always negative. Moreover, simplifying the inequality $\frac{b}{a+b} > 1$, it is easy to see that we get $a_{11} - a_{21} < \sigma_1^2$. Since $\sigma_2^2 < a_{22} - a_{12}$, this brings us to the case I(i), and so we may disregard this inequality.

Recalling the theorem of Kushner [14], we will determine a stochastic Lyapunov function for each of the inequalities stated above. Clearly, by the boundary absorption property, the initial condition is strictly in the unit interval.

For I(i), we need an f where $f \geq 0$, $f' \leq 0$ and $f'' \leq 0$. We see that $f(y) = 1 - y$ holds these characteristics. The theorem in Kushner tells us that the process converges to the set $\{0, 1\}$.

For II(i), we need an f where $f \geq 0$, $f' \geq 0$ and $f'' \leq 0$. We see that $f(y) = y$ will work, and like the previous game, we have convergence to the set $\{0, 1\}$.

For the I(ii) case, a nonnegative f in which $f'(y) \geq 0$ for $y \in (0, b/(a+b))$ and $f'(y) \leq 0$ for $y \in (b/(a+b), 1)$, and clearly $f'' \leq 0$ is needed. Defining $f(y) := 16 - (y - b/(a+b))^2$ we get these characteristics and so the process converges to the set $\{0, 1\}$.

As for II(ii), I have yet to find a function that is in $D(\mathcal{A})$ to apply to this case. Considering the results of Fudenberg and Harris, $\left(\text{if } a_{11} - a_{21} < (\sigma_1^2 - \sigma_2^2)/2 \text{ and } a_{22} - a_{12} < (\sigma_2^2 - \sigma_1^2)/2 \text{ then } P\left(\liminf_{t\to\infty} s_1(t) = 0\right) = P\left(\limsup_{t\to\infty} s_1(t) = 1\right) = 1 \right)$, we shouldn't be able to find such a function.

Since the outcomes of the stochastic Lyapunov method do not give us probabilities for 0 and 1, in order to be more precise we need to determine the invariant (ergodic) measure of the process. We will begin by defining the new process $\hat{s}(t) = p(s(t))$ (for $p(x)$ defined at the beginning of the section), $c_1 := p(0)$ and $c_2 := p(1)$. So $\hat{s}(t) = c_1$ when $s(t) = 0$ and $\hat{s}(t) = c_2$ when $s(t) = 1$. Furthermore, define $f(x) = p^{-1}(x)$, and $\hat{b}(x) = \sigma p(f(x)) f(x)(1 - f(x))$. Then following theorem gives us conditions to compute the invariant measure.

Theorem 3.2.1. *([19], page 54) If $-\infty < c_1 < c_2 \leq \infty$ then a solution to $\hat{s}_1(t)$ exists for all t iff $\int_{c_1}^{c} \hat{\beta}^{-2}(y)dy = \infty$ and $c_2 + \int_{c}^{c_2} \hat{\beta}^{-2}(y)dy = \infty$ for some $c \in (c_1, c_2)$.*

Under the latter conditions, if $c_2 = \infty$ then $P_x\left(\lim_{t\to\infty} \hat{s}_1(t) = c_1\right) = 1$. Moreover, if $c_2 < \infty$ then $P_x\left(\lim_{t\to\infty} \hat{s}_1(t) = c_1\right) = \frac{c_2}{c_2 - c_1}$ and $P_x\left(\lim_{t\to\infty} \hat{s}_1(t) = c_2\right) = \frac{c_1}{c_2 - c_1}$.

The stochastic Lyapunov method gives us the latter conditions and thus we are able to recreate partial results of Fudenberg and Harris. What we have shown is if we have

- $a_{11} - a_{21} > (\sigma_1^2 - \sigma_2^2)/2$ and $a_{22} - a_{12} < (\sigma_2^2 - \sigma_1^2)/2$, or
- $a_{11} - a_{21} < (\sigma_1^2 - \sigma_2^2)/2$ and $a_{22} - a_{12} > (\sigma_2^2 - \sigma_1^2)/2$, or
- $a_{11} - a_{21} > (\sigma_1^2 - \sigma_2^2)/2$ and $a_{22} - a_{12} > (\sigma_2^2 - \sigma_1^2)/2$

then we need

- $\sigma_2^2 < a_{22} - a_{12}$ and $\sigma_1^2 > a_{11} - a_{21}$, or
- $\sigma_2^2 > a_{22} - a_{12}$ and $\sigma_1^2 < a_{11} - a_{21}$, or
- $\sigma_2^2 < a_{22} - a_{12}$ and $\sigma_1^2 < a_{11} - a_{21}$

to hold as well in order to describe the invariant measures.

Remark 3.2.1. *Even though we are able to compute the invariant measure for all of the possible cases [19], we wanted to see how far the stochastic Lyapunov method could take the analysis of the model.*

3.3 Applications to the Model

The analysis of our model will be done in the same manner as in the previous section. Define $\mathcal{A}_J$ under the supremum norm, to be our generator. Hence $\mathcal{A}_J$ takes the form

$$\begin{aligned}
\mathcal{A}_J(f)(y) =& \lim_{t \searrow 0} \frac{\mathbb{E}[f(s(t)+y)] - f(y)}{t} \\
=& \Bigg\{ y(1-y)\left[(a_{11}-a_{21})y + (a_{12}-a_{22})(1-y) + \sigma_2^2(1-y) - \sigma_1^2 y\right] \\
&+ \int_{\mathbb{R}} \left(\frac{y[1-y][h_1(x)-h_2(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} + y(1-y)(h_2(x)-h_1(x)) \right) \nu(dx) \Bigg\} f'(y) \\
&+ \frac{1}{2}\sigma^2 y^2 (1-y)^2 f''(y) \\
&+ \int_{\mathbb{R}} \Bigg\{ f\left(y + \frac{y[1-y][h_1(x)-h_2(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} \right) - f(y) \\
&\quad - \frac{y[1-y][h_1(x)-h_2(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} f'(y) \Bigg\} \nu(dx)
\end{aligned}$$

$$
\begin{aligned}
=&\Bigg\{y(1-y)\left[(a_{11}-a_{21})y+(a_{12}-a_{22})(1-y)+\sigma_2^2(1-y)-\sigma_1^2 y\right] \\
&+\int_{\mathbb{R}} y(1-y)(h_2(x)-h_1(x))\nu(dx)\Bigg\}f'(y) \\
&+\frac{1}{2}\sigma^2 y^2(1-y)^2 f''(y) \\
&+\int_{\mathbb{R}}\left\{f\left(y+\frac{y[1-y][h_1(x)-h_2(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right)-f(y)\right\}\nu(dx) \\
=&\ y(1-y)\Bigg[\left(a_{12}-a_{22}+\sigma_2^2+\int_B (h_2(x)-h_1(x))\nu(dx)\right) \\
&+\{(a_{11}+a_{22})-(a_{12}+a_{21}+\sigma_1^2+\sigma_2^2)\}yBigg]f'(y) \\
&+\frac{1}{2}\sigma^2 y^2(1-y)^2 f''(y) \\
&+\int_{\mathbb{R}}\left\{f\left(\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right)-f(y)\right\}\nu(dx) \\
:=&-k(y).
\end{aligned}
\tag{3.3}
$$

We will show that $C_b^2([0,1]) \subset D(\mathcal{A}_J)$, but before we are able to do this we need to ascertain some characteristics of $\dfrac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}$.

Claim 3.3.1. *For every $y \in (0,1)$ and $x \in \mathbb{R}$, $y[h_1(x)-h_2(x)]+1+h_2(x) > 0$.*

Proof. We will use proof by contradiction. Suppose that there exists an x and y such that $y[h_1(x)-h_2(x)]+1+h_2(x) \leq 0$. So we will work with the inequality $y[h_1(x)-h_2(x)] \leq -(1+h_2(x))$. First let us suppose that $h_1(x)-h_2(x) > 0$. Thus

$$
y \leq \frac{-(1+h_2(x))}{h_1(x)-h_2(x)} \implies 0 \leq \frac{-(1+h_2(x))}{h_1(x)-h_2(x)} \implies 0 \leq -(1+h_2(x)) \implies h_2(x) \leq -1,
$$

which is a contradiction from our assumption on $h_2(x)$.

Now let us suppose that $h_1(x)-h_2(x) < 0$. Hence $y \geq \dfrac{-(1+h_2(x))}{h_1(x)-h_2(x)}$, which implies $1 \geq \dfrac{-(1+h_2(x))}{h_1(x)-h_2(x)}$. The rest is similar to that above. $\square$

Remark 3.3.1. *Since $\Upsilon(y) := \dfrac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}$ is continuous by the claim above with $\Upsilon(y)=0$ and $\Upsilon(y)=1$ if and only if $y=0$ and $y=1$, respectively, it is clear that $0 \leq \Upsilon(y) \leq 1$ for $y \in [0,1]$.*

Proposition 3.3.1. $C_b^2([0,1]) \subset D(\mathcal{A}_J)$.

Proof. Since our space is compact we will apply Theorem 3.5 in [22]. So it must be shown that:

(a) $D(\mathcal{A}_J)$ is dense in $C([0,1])$;

(b) there exists an open, dense subset of $[0,1]$, say Q_0, such that if $u \in D(\mathcal{A}_J)$ takes a positive maximum at a point $q_0 \in Q_0$, then $\mathcal{A}_J u(q_0) \leq 0$.

By the structure of $\mathcal{A}_J$, we have that $\mathcal{A}_J f(0) = \mathcal{A}_J f(1) = 0$ for every $f \in C^\infty([0,1])$. Thus, it is clear that $C^\infty([0,1]) \subset D(\mathcal{A}_J)$. Moreover, it is well known that $C^\infty([0,1])$ is dense in $C([0,1])$. Therefore, (a) is shown.

To show (b), take $Q_0 \subset (0,1)$, open and dense, and a $u \in D(\mathcal{A}_J)$ such that for $q_0 \in Q_0$, $u(q_0)$ is a positive maximum of u. Since q_0 is a positive maximum, $u'(q_0) = 0$ and $u''(q_0) \leq 0$. Moreover, by the remark above, we have that $u\left(q_0 + \dfrac{q_0[1-q_0][h_1(x)-h_2(x)]}{q_0[h_1(x)-h_2(x)]+1+h_2(x)}\right) - u(q_0) \leq 0$. Thus $\mathcal{A}_J u(q_0) \leq 0$ and we are done. □

Just as in the previous section, we will determine conditions for the sign of $\zeta_J(y) := -b_J + \{a+b\}y$, where $b_J := a_{22} - a_{12} - \sigma_2^2 + \int_B \Big(h_1(x) - h_2(x)\Big)\nu(dx)$, and a and b are defined as in the previous section. We also need the sign of

$$\int_{\mathbb{R}} \left\{ f\left(\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right) - f(y) \right\} \nu(dx)$$

in order to apply a stochastic Lyapunov function without much complication.

The following is a list of inequalities associated with the sign of $\zeta_J(y)$:

(A) $\zeta_J(y) \geq 0$ for all $y \in (0,1)$ implies $a_{22} - a_{12} < \sigma_2^2 + \int_{\mathbb{R}} \Big(h_2(x) - h_1(x)\Big)\nu(dx)$ and $a_{11} - a_{21} > \sigma_1^2 + \int_{\mathbb{R}} \Big(h_1(x) - h_2(x)\Big)\nu(dx)$;

(B) $\zeta_J(y) \leq 0$ for all $y \in (0,1)$ implies that $a_{22} - a_{12} > \sigma_2^2 + \int_{\mathbb{R}} \Big(h_2(x) - h_1(x)\Big)\nu(dx)$ and $a_{11} - a_{21} < \sigma_1^2 + \int_{\mathbb{R}} \Big(h_1(x) - h_2(x)\Big)\nu(dx)$;

(C) the sign change of $\zeta_J(y)$ from negative to positive when $y \in (0, b_J/(a+b))$ and $y \in (b_J/(a+b), 1)$ respectively, gives the inequalities $a_{22} - a_{12} > \sigma_2^2 + \int_{\mathbb{R}} \Big(h_2(x) - h_1(x)\Big)\nu(dx)$ with $a_{11} - a_{21} > \sigma_1^2 + \int_{\mathbb{R}} \Big(h_1(x) - h_2(x)\Big)\nu(dx)$;

(D) and lastly, $\zeta_J(y)$ is positive for $y \in (0, b_J/(a+b))$ and negative otherwise tells us that $a_{22} - a_{12} < \sigma_2^2 + \int_B \Big(h_2(x) - h_1(x)\Big)\nu(dx)$ along with $a_{11} - a_{21} < \sigma_1^2 + \int_{\mathbb{R}} \Big(h_1(x) - h_2(x)\Big)\nu(dx)$.

Before we begin let us fix an arbitrary $y \in (0,1)$ and make some observations about when the inequalities below hold.

Observation 3.3.1. *We have that*

$$\begin{aligned}\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} > y &\iff y+yh_1(x) > y^2h_1(x)-y^2h_2(x)+y+yh_2(x) \\ &\iff 0 > y^2h_1(x)-y^2h_2(x)-yh_1(x)+yh_2(x) \\ &\iff 0 > (y^2-y)(h_1(x)-h_2(x)),\end{aligned}$$

and since $y^2-y<0$ for every $y\in(0,1)$, we have that $h_1(x)-h_2(x)>0$, hence $h_1(x)>h_2(x)$.

Observation 3.3.2. *Reversing the inequality above, we see that*

$$\begin{aligned}\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} < y &\iff y+yh_1(x) < y^2h_1(x)-y^2h_2(x)+y+yh_2(x) \\ &\iff 0 < y^2h_1(x)-y^2h_2(x)-yh_1(x)+yh_2(x) \\ &\iff 0 < (y^2-y)(h_1(x)-h_2(x)),\end{aligned}$$

and since $y^2-y<0$ for any $y\in(0,1)$, we have that $h_1(x)-h_2(x)<0$, i.e., $h_2(x)>h_1(x)$.

Observation 3.3.3. *Finally,*

$$\begin{aligned}\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} = y &\iff y+yh_1(x) = y^2h_1(x)-y^2h_2(x)+y+yh_2(x) \\ &\iff (1-y)h_1(x)=(1-y)h_2(x) \\ &\iff h_1(x)=h_2(x).\end{aligned}$$

Definition 3.3.1. *We will say that the i^{th} subpopulation has jump dominance if $h_i(x)>h_j(x)$ for all $j\neq i$ and for all $x\in\mathbb{R}$.*

We will now analyze the jump dominance. There are many cases in which just assuming jump dominance is not enough for a smooth application and so we will add more assumptions of the process. Recall that $\zeta(y)=-b+\{a+b\}y$ and that the initial condition is strictly in the unit interval. Details shown in the previous section are assumed to be known.

1. We will first consider 1^{st} subpopulation jump dominance. So for each jump, 1^{st} subpopulation growth fairs better than 2^{nd} subpopulation growth.

(a) We will assume inequality (A) and apply the Lyapunov function $f(y) = 1 - y$. Thus we have that

$$f\left(\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right) - f(y) = y - \left(\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right)$$

is negative by observation (3.3.1). Therefore $s(t) \to \{0,1\}$ a.s.

(b) We will now determine how this dominance affects the inequality (B). Applying $f(y) = y$, and assuming $\zeta(y) \leq 0$ for all $y \in (0,1)$ and $h_2(x) \geq 0$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
-k(y) =& y(1-y)\zeta_J(y) + \int_{\mathbb{R}} \left[\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} - y\right]\nu(dx) \\
=& y(1-y)\zeta(y) + \int_{\mathbb{R}} \left[y(1-y)(h_2(x)-h_1(x)) + \frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} - y\right]\nu(dx) \\
=& y(1-y)\left(\zeta(y) + \int_{\mathbb{R}} \left[(h_2(x)-h_1(x)) + \frac{yh_1(x) - y^2h_1(x) + y^2h_2(x) - yh_2(x)}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right]\nu(dx)\right) \\
=& y(1-y)\left(\zeta(y) + \int_{\mathbb{R}} (h_2(x)-h_1(x))\left[1 + \frac{-1}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right]\nu(dx)\right) \\
=& -y(1-y)\left(-\zeta(y) + \int_{\mathbb{R}} (h_1(x)-h_2(x))\left[\frac{yh_1(x)+(1-y)h_2(x)}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right]\nu(dx)\right).
\end{aligned} \tag{3.4}$$

which is negative. Therefore $s(t) \to \{0,1\}$ a.s.

(c) Continuing with inequality (B), we will apply $f(y) = 1 - y$ and assume that $\zeta(y) \geq 0$ for all $y \in (0,1)$ and $h_1(x) \leq 0$ for all $x \in \mathbb{R}$. Hence

$$\begin{aligned}
-k(y) =& y(1-y)\zeta_J(y)(-1) + \int_{\mathbb{R}} \left[y - \frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right]\nu(dx) \\
=& y(1-y)\left(-\zeta(y) + \int_{\mathbb{R}} \left[(h_1(x)-h_2(x)) + \frac{h_2(x)-h_1(x)}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right]\nu(dx)\right) \\
=& y(1-y)\left(-\zeta(y) + \int_{\mathbb{R}} (h_1(x)-h_2(x))\left[1 + \frac{-1}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right]\nu(dx)\right) \\
=& -y(1-y)\left(\zeta(y) + \int_{\mathbb{R}} (h_2(x)-h_1(x))\left[\frac{yh_1(x)+(1-y)h_2(x)}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right]\nu(dx)\right),
\end{aligned} \tag{3.5}$$

which is negative by our assumptions and so $s(t) \to \{0,1\}$ a.s.

2. Next we will assume the 2^{nd} subpopulation is jump dominant.

(a) Taking $f(y) = y$, since $f\left(\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right) - f(y) = \left(\frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)}\right) - y$ is negative by observation (3.3.2) and hence $s(t) \to \{0,1\}$ a.s.

(b) Considering inequality (A), we will make the assumption $h_1(x) \geq 0$ for all $x \in \mathbb{R}$. Taking $f(y) = 1 - y$, we determine that

$$
\begin{aligned}
-k(y) =& -y(1-y)\zeta_J(y) + \int_{\mathbb{R}} \left[y - \frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} \right] \nu(dx) \\
=& -y(1-y)\left(\zeta(y) + \int_{\mathbb{R}} (h_2(x)-h_1(x)) \left[1 + \frac{-1}{y[h_1(x)-h_2(x)]+1+h_2(x)} \right] \nu(dx) \right) \\
=& -y(1-y)\left(\zeta(y) + \int_{\mathbb{R}} (h_2(x)-h_1(x)) \left[\frac{yh_1(x)+(1-y)h_2(x)}{y[h_1(x)-h_2(x)]+1+h_2(x)} \right] \nu(dx) \right),
\end{aligned}
\tag{3.6}
$$

which is negative by our assumption. Thus $s(t) \to \{0,1\}$ a.s.

(c) Furthermore, for inequality (A), we will assume that $\zeta(y) \geq 0$ for all $y \in (0,1)$ and $h_1(x) \leq 0$ for all $x \in B$. Taking $f(y) = 1-y$ yeilds

$$
\begin{aligned}
-k(y) =& y(1-y)\zeta_J(y)(-1) + \int_{\mathbb{R}} \left[y - \frac{y[1+h_1(x)]}{y[h_1(x)-h_2(x)]+1+h_2(x)} \right] \nu(dx) \\
=& y(1-y)\left(-\zeta(y) + \int_{\mathbb{R}} \left[(h_1(x)-h_2(x)) + \frac{h_2(x)-h_1(x)}{y[h_1(x)-h_2(x)]+1+h_2(x)} \right] \nu(dx) \right) \\
=& y(1-y)\left(-\zeta(y) + \int_{\mathbb{R}} (h_1(x)-h_2(x)) \left[1 + \frac{-1}{y[h_1(x)-h_2(x)]+1+h_2(x)} \right] \nu(dx) \right) \\
=& -y(1-y)\left(\zeta(y) + \int_{\mathbb{R}} (h_2(x)-h_1(x)) \left[\frac{yh_1(x)+(1-y)h_2(x)}{y[h_1(x)-h_2(x)]+1+h_2(x)} \right] \nu(dx) \right),
\end{aligned}
\tag{3.7}
$$

which is negative by our assumptions and therefore $s(t) \to \{0,1\}$ a.s.

Note 3.3.1. *What was shown is that under certain assumptions, we have that our evolutionary process converges to either endpoint, but without the probabilities for 0 and 1. As such, this technique falls short of our goal. However, we have conditions for the existence of an invariant measure.*

We will continue with this technique and assume that $h_1(x)$ and $h_2(x)$ are constants, which we will denote as h_1 and h_2. We will only apply the stochastic Lyapunov functions $f(y) = 1-y$ and $f(y) = y$ and determine conditions for which an invariant measure exists. Only using these functions simplifies the analysis, however, as will be seen below, there are cases which we will be unable to say anything about the process.

Applying $f(y) = 1 - y$ we get that

$$
\begin{aligned}
k(y) =& y(1-y)\zeta_J(y)(-1) + \int_{\mathbb{R}} \left[y - \frac{y[1+h_1]}{y[h_1-h_2]+1+h_2} \right] \nu(dx) \\
=& y(1-y) \left(-\zeta(y) + \int_{\mathbb{R}} \left[(h_1-h_2) + \frac{h_2-h_1}{y[h_1-h_2]+1+h_2} \right] \nu(dx) \right) \\
=& y(1-y) \left(-\zeta(y) + \int_{\mathbb{R}} (h_1-h_2) \left[1 + \frac{-1}{y[h_1-h_2]+1+h_2} \right] \nu(dx) \right) \\
=& -y(1-y) \left(\zeta(y) + \nu(\mathbb{R})(h_2-h_1) \left[\frac{y[h_1-h_2]+h_2}{y[h_1-h_2]+1+h_2} \right] \right),
\end{aligned}
\tag{3.8}
$$

and so we must find conditions when

$$
\begin{aligned}
& \zeta(y) + \nu(\mathbb{R})(h_2-h_1) \left[\frac{y[h_1-h_2]+h_2}{y[h_1-h_2]+1+h_2} \right] \geq 0 \\
\iff & ([h_1-h_2]y+1+h_2)(-b+[a+b]y) - \nu(\mathbb{R})[h_1-h_2]^2 y + \nu(\mathbb{R})[h_2-h_1]h_2 \geq 0.
\end{aligned}
$$

Simplifying, we see that

$$
\begin{aligned}
& (h_1-h_2)(a+b)y^2 \\
& +(\ a[h_2+1] + b[-h_1+2h_2+1] - \nu(\mathbb{R})[h_1-h_2]^2 \ )y \\
& +(\ -b[h_2+1] + \nu(\mathbb{R})[h_2-h_1]h_2 \ ). \\
:=& a'y^2 + b'y + c'. \\
:=& \varphi(y)
\end{aligned}
\tag{3.9}
$$

The quadratic equation yields the roots

$$
\frac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'} \text{ and } \frac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}.
$$

Our method for determining convergence is to first assign a sign for each coefficient then check the size of each root. Before we begin, notice that we need the constant term to always be positive, hence we will assume that

$$
b < \frac{\nu(\mathbb{R})[h_2-h_1]h_2}{h_2+1}
$$

throughout the rest of the analysis.

1. Clearly if $a', b' \geq 0$, we would have almost everywhere convergence to the set $\{0, 1\}$.
2. Assume the inequalities $a' > 0$ and $b' < 0$. When $(b')^2 - 4a'c' \geq 0$, since $\dfrac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'} \geq \dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}$,

we only need to check when

$$
\begin{aligned}
&\frac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'} \geq 1\\
\iff & -b' - \sqrt{(b')^2 - 4a'c'} \geq 2a'\\
\iff & -\sqrt{(b')^2 - 4a'c'} \geq 2a' + b'\\
\iff & (b')^2 - 4a'c' \leq 4(a')^2 + 4a'b' + (b')^2\\
\iff & 0 \leq a' + b' + c'\\
\iff & \frac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1} \leq a.
\end{aligned}
$$

Hence the process converges to the set $\{0, 1\}$ a.s. with this inequality. Moreover, when $(b')^2 - 4a'c' < 0$, which is if and only if $b^2(1 + h_1)^2 + a^2(1 + h_2)^2 + (h_1 - h_2)^4\nu(\mathbb{R})^2 < -2b(-1 + h_1)(h_1 - h_2)^2\nu(\mathbb{R}) - 2a(b(1 + h_1)(1 + h_2) + (h_1 - h_2)^2(-1 + h_2)\nu(\mathbb{R}))$, we also get convergence.

3. We will now consider when $a', b' < 0$. Since $\frac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'} \leq 0$, we only need

$$
\begin{aligned}
&\frac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'} \geq 1\\
\iff & -\sqrt{(b')^2 - 4a'c'} \leq 2a' + b'\\
\iff & (b')^2 - 4a'c' \geq 4(a')^2 + 4a'b' + (b')^2\\
\iff & a' + b' + c' \geq 0\\
\iff & a \geq \frac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}.
\end{aligned}
$$

Therefore, if the last inequality holds, we will have almost sure convergence to the set $\{0, 1\}$.

4. Lastly, we will work with the inequalities $a' < 0$ and $b' > 0$. Since

$$
\frac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'} \leq 0 \iff c' \geq 0,
$$

we are left with

$$
\frac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'} \geq 1 \iff -\sqrt{(b')^2 - 4a'c'} \leq 2a' + b'.
$$

From here we have two possibilities: $|2a' + b'| \leq \sqrt{(b')^2 - 4a'c'}$, which is if and only if $a \geq \frac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$; and $|2a' + b'| \geq \sqrt{(b')^2 - 4a'c'}$, and since this would imply that $2a' + b' > 0$, we have

$$
2a' + b' \geq \sqrt{(b')^2 - 4a'c'} \iff b' \geq \sqrt{(b')^2 - 4a'c'} - 2a' > \sqrt{(b')^2} - 2a' = b' - 2a' > b',
$$

which is a contradiction Thus for $a \geq \frac{\nu(B)[h_1 - h_2]h_1}{h_1 + 1}$, we have almost everywhere convergence to the set

$\{0,1\}$.

Next we will find criteria so that the process converges almost everywhere to 0. Applying $f(y)=y$ we get

$$\begin{aligned} k(y) =& y(1-y)\zeta_J(y) + \int_{\mathbb{R}} \left[\frac{y[1+h_1]}{y[h_1-h_2]+1+h_2} - y \right] \nu(dx) \\ =& -y(1-y)\left(-\zeta(y) + \nu(\mathbb{R})(h_1-h_2) \left[\frac{y[h_1-h_2]+h_2}{y[h_1-h_2]+1+h_2} \right] \right), \end{aligned} \tag{3.10}$$

and so, as above, we need to determine when

$$\begin{aligned} &-\zeta(y) + \nu(\mathbb{R})(h_1-h_2) \left[\frac{y[h_1-h_2]+h_2(x)}{y[h_1-h_2]+1+h_2} \right] \geq 0 \\ \iff& ([h_1-h_2]y+1+h_2)(b-[a+b]y) + \nu(\mathbb{R})[h_1-h_2]^2 y + \nu(\mathbb{R})[h_1-h_2]h_2 \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} &(h_2-h_1)(a+b)y^2 \\ &+(\ a[-h_2-1] + b[h_1-2h_2-1] + \nu(\mathbb{R})[h_1-h_2]^2\ )y \\ &+(\ b[h_2+1] + \nu(\mathbb{R})[h_1-h_2]h_2\ ). \\ &= -(a'y^2+b'y+c'). \\ &= -\varphi(y) \end{aligned} \tag{3.11}$$

Thus we have the same roots,

$$\frac{-b'+\sqrt{(b')^2-4a'c'}}{2a'} \text{ and } \frac{-b'-\sqrt{(b')^2-4a'c'}}{2a'},$$

and will find conditions when $a'y^2+b'y+c' \leq 0$ for $y \in [0,1]$. Therefore, we will use the same methods and omit details. Here we need the constant term to always be negative and hence we will assume that

$$b > \frac{\nu(\mathbb{R})[h_2-h_1]h_2}{h_2+1}.$$

A. If $a', b' \leq 0$, we would have almost everywhere convergence to the set $\{0,1\}$.

B. Assume the inequalities $a' > 0$ and $b' < 0$. Since $\frac{-b'-\sqrt{(b')^2-4a'c'}}{2a'} \leq 0$ we only need to check when $\frac{-b'+\sqrt{(b')^2-4a'c'}}{2a'} \geq 1 \iff \sqrt{(b')^2-4a'c'} \geq 2a'+b'$. Reasoning as above, we have that $\sqrt{(b')^2-4a'c'} \geq |2a'+b'|$ and hence $a \leq \frac{\nu(B)[h_1-h_2]h_1}{h_1+1}$. Hence, under this inequality, the process converges almost surely to the set $\{0,1\}$.

C. We will now consider when $a', b' > 0$. Since $\dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'} \leq 0$, we only need

$$\frac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'} \geq 1 \iff a \leq \frac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}.$$

Therefore, if the last inequality holds, we will have almost sure convergence to the set $\{0, 1\}$.

D. Lastly, we will work with the inequalities $a' < 0$ and $b' > 0$. If $(b')^2 - 4a'c' < 0$, we are done. Assume $(b')^2 - 4a'c' \geq 0$. Then since $\dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'} \geq 0$ we are left with

$$\frac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'} \geq 1 \iff a \leq \frac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}.$$

Thus we have almost everywhere convergence to the set $\{0, 1\}$.

We will now look at the rest of the cases.

I. Suppose that $a', b' > 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a > \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$. The last inequality implies that the positive root $\dfrac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'} < 1$. Thus $-\varphi(y)$ is positive for $y \in \left[0, \dfrac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'}\right)$ and negative for $y \in \left(\dfrac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'}, 1\right]$.

II. If $a' < 0$, $b' > 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $(b')^2 - 4a'c' \geq 0$ with $a > \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then, since $\dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'} < 1 \implies a < \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$, we have that $-\varphi(y)$ is positive for $y \in \left[0, \dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}\right)$ and negative for $y \in \left(\dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}, 1\right]$.

III. If $a' > 0$, $b' < 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a > \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then $-\varphi(y)$ is positive for $y \in \left[0, \dfrac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'}\right)$ and negative for $y \in \left(\dfrac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'}, 1\right]$.

IV. If $a' > 0$, $b' < 0$ and $b < \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ with $(b')^2 - 4a'c' \geq 0$ and $a < \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$, then, since $\dfrac{-b' + \sqrt{(b')^2 - 4a'c'}}{2a'} < 1 \implies a > \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$, this implies that $\varphi(y)$ is positive for $y \in \left[0, \dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}\right)$ and negative for $y \in \left(\dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}, 1\right]$.

V. If $a', b' < 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a < \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then $\varphi(y)$ is positive for $y \in \left[0, \dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}\right)$ and negative for $y \in \left(\dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}, 1\right]$.

VI. If $a' < 0$, $b' > 0$, $b < \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a < \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then, yet again, $\varphi(y)$ is positive for $y \in \left[0, \dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}\right)$ and negative for $y \in \left(\dfrac{-b' - \sqrt{(b')^2 - 4a'c'}}{2a'}, 1\right]$.

Collecting everything we showed above, we have the criteria:

(i.) If $a', b' \geq 0$ and $b < \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ then $s(t) \to \{0, 1\}$ a.s.

(ii.) If $a' > 0$, $b' < 0$ and $b < \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ with $(b')^2 - 4a'c' \geq 0$ and $a \geq \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ or $(b')^2 - 4a'c' \leq 0$ then $s(t) \to \{0, 1\}$ a.s.

(iii.) If $a', b' < 0$, $b < \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a \geq \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then $s(t) \to \{0, 1\}$ a.s.

(iv.) If $a' < 0$, $b' > 0$, $b < \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a \geq \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then $s(t) \to \{0, 1\}$ a.s.

(v.) If $a', b' \leq 0$ and $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ then $s(t) \to \{0, 1\}$ a.s.

(vi.) If $a' > 0$ and $b' < 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a \leq \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then $s(t) \to \{0, 1\}$ a.s.

(vii.) If $a', b' > 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a \leq \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then $s(t) \to \{0, 1\}$ a.s.

(viii.) If $a' < 0$, $b' > 0$ and $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ with $(b')^2 - 4a'c' < 0$ or $(b')^2 - 4a'c' \geq 0$ and $a \leq \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$ then $s(t) \to \{0, 1\}$ a.s.

The rest of the cases below are inequalities for which we are unable to give existence of an invariant measure.

1. If $a', b' > 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a > \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$.
2. If $a' < 0$, $b' > 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $(b')^2 - 4a'c' \geq 0$ with $a > \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$.
3. If $a' > 0$, $b' < 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a > \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$.
4. If $a' > 0$, $b' < 0$ and $b < \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ with $(b')^2 - 4a'c' \geq 0$ and $a < \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$.
5. If $a', b' < 0$, $b > \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a < \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$.
6. If $a' < 0$, $b' > 0$, $b < \dfrac{\nu(\mathbb{R})[h_2 - h_1]h_2}{h_2 + 1}$ and $a < \dfrac{\nu(\mathbb{R})[h_1 - h_2]h_1}{h_1 + 1}$.

Chapter 4

Analysis of the Two Strategy Model

4.1 Calculation of the Approximated Invariant Measure

The stochastic Lyapunov method fell short of giving us exact information about the long run behavior the process. The optimal result would be a theorem similar to that of Fudenberg and Harris [7] in which after verifying the inequalities of the payoffs with respect to a combination of the white noise gives us this characteristic. After a reasonable assumption, we will apply results given in [23] and [1] to give an adjustment of the inequalities given in Fudenberg and Harris [7]. These inequalities are an approximation of the actual result since it is very difficult to completely solve.

Define

$$\tilde{\alpha}(y) = y(1-y)\left[\left(a_{12} - a_{22} + \sigma_2^2 + \int_{\mathbb{R}} (h_2(x) - h_1(x))\nu(dx)\right) + \{(a_{11} - a_{21} - \sigma_1^2) + (a_{22} - a_{12} - \sigma_2^2)\}y\right],$$

$$\tilde{\beta}(y) = \frac{\sigma^2}{2} y^2(1-y)^2,$$

and

$$y + \gamma(y,x) = \frac{y[1+h_1(x)]}{y[h_1(x) - h_2(x)] + 1 + h_2(x)}.$$

Moreover, for some $0 < y_1 < y_2 < 1$, define $\tau_{y_1 y_2}(y_0) = \inf_{T \geq 0}\{s(T) \notin (y_1, y_1) | s(0) = y_0\}$, $\pi_{y_2;y_1}(y_0) = P\Big(s(\tau_{y_1 y_2}(y_0)) \geq y_2\Big)$, and $\pi_{y_1;y_2}(y_0) = \Big(s(\tau_{y_1 y_2}(y_0)) \leq y_1\Big)$.

Consider an integro-differential equation of the form

$$\tilde{\alpha}(y)u'(y) + \tilde{\beta}(y)u''(y) + \int_{\mathbb{R}} [u(y + \gamma(y,x)) - u(y)]\,\nu(dx) = 0 \quad \text{for } y \in (a,b), \tag{4.1}$$

with the conditions $u(y) = 0$ for $y \in [0, y_1]$, and $u(y) = 1$ for $y \in [y_2, 1]$. The papers of [23] and [1] tell us that solving the integro-differential equation above will give us $\pi_{y_2;y_1}(y_0)$, (interchanging the initial conditions will give $\pi_{y_1;y_2}(y_0)$.) However, in order to apply these theorems we need to verify the condition that $E[\tau^n_{y_1 y_2}(y_0)]$ is finite for every n. Theorem 5.1 in the Chapter 5 §1, ensures us that this property holds.

Remark 4.1.1. *We should note here that the result in Tuckwell [23] is for a jump-diffusion with a Poisson measure and not the compensated Poisson measure. However, Tuckwell's proof is based off a result in Gihman and Skorohod [8], in which the authors give an equality for the transition probability for a jump-diffusion with a compensated Poisson measure (**Part II** Chapter 2 §9). A simple adjustment in the first order coefficient will give the equivalent conclusions.*

Solving this integro-differential equation is a very difficult task, and so we will construct a way to approximate the solution. First we will assume that $h_1(x) = h_2(x) + \epsilon$, for a small, arbitrary $\epsilon \in \mathbb{R}$. Next, we will turn the integral into a Taylor series, using ϵ as the variable, grouping the higher order terms into an error term. Note that we will use the function f, instead of u, to find the solution, since normalizing f and considering the initial conditions will determine u.

By our assumption we have that the integral difference term is

$$\int_{\mathbb{R}} \left[f\left(\frac{y[1+h_2(x)+\epsilon]}{\epsilon y + 1 + h_2(x)} \right) - f(y) \right] \nu(dx) := F(\epsilon).$$

Clearly $F(0) = 0$,

$$\begin{aligned}
F'(0) &= \int_{\mathbb{R}} f'\left(\frac{y[1+h_2(x)+\epsilon]}{\epsilon y + 1 + h_2(x)} \right) \left(\frac{y[1+h_2(x)+\epsilon]}{\epsilon y + 1 + h_2(x)} \right)' \nu(dx) \Bigg|_{\epsilon=0} \\
&= \int_{\mathbb{R}} f'\left(\frac{y[1+h_2(x)+\epsilon]}{\epsilon y + 1 + h_2(x)} \right) \left(\frac{y(1-y)[1+h_2(x)]}{(\epsilon y + 1 + h_2(x))^2} \right) \nu(dx) \Bigg|_{\epsilon=0} \\
&= y(1-y)f'(y) \int_{\mathbb{R}} \frac{1}{1+h_2(x)} \nu(dx) = C_1 y(1-y) f'(y),
\end{aligned}$$

and

$$\begin{aligned}
F''(0) = &\int_{\mathbb{R}} \left\{ f''\left(\frac{y[1+h_2(x)+\epsilon]}{\epsilon y + 1 + h_2(x)} \right) \left(\frac{y(1-y)[1+h_2(x)]}{(\epsilon y + 1 + h_2(x))^2} \right)^2 \right. \\
&\left. + f'\left(\frac{y[1+h_2(x)+\epsilon]}{\epsilon y + 1 + h_2(x)} \right) \left(\frac{-2y^2(1-y)[1+h_2(x)]}{(\epsilon y + 1 + h_2(x))^3} \right) \right\} \nu(dx) \Bigg|_{\epsilon=0} \\
&= -2y^2(1-y)f'(y) \int_{\mathbb{R}} \frac{1}{[1+h_2(x)]^2} \nu(dx) + y^2(1-y)^2 f''(y) \int_{\mathbb{R}} \frac{1}{[1+h_2(x)]^2} \nu(dx) \\
&= -2C_2 y^2(1-y) f'(y) + C_2 y^2 (1-y)^2 f''(y),
\end{aligned}$$

where $C_1 := \displaystyle\int_{\mathbb{R}} \frac{1}{1+h_2(x)} \nu(dx)$ and $C_2 := \displaystyle\int_{\mathbb{R}} \frac{1}{[1+h_2(x)]^2} \nu(dx)$. Thus

$$F(\epsilon) = C_1 y(1-y) f'(y)\epsilon + (-2C_2 y^2(1-y) f'(y) + C_2 y^2(1-y)^2 f''(y))\epsilon^2/2 + O(\epsilon^3).$$

Excluding the error term, equation (4.1) becomes the second order differential equation

$$\tilde{\alpha}_\epsilon(y) f'(y) + \tilde{\beta}_\epsilon(y) f''(y) = 0,$$

where

$$\tilde{\alpha}_\epsilon(y) := y(1-y)\left[\left(a_{12} - a_{22} + \sigma_2^2 + \epsilon(C_1 - \nu(\mathbb{R}))\right) + \{(a_{11} - a_{21} - \sigma_1^2) + (a_{22} - a_{12} - \sigma_2^2) - \epsilon^2 C_2\}y\right]$$

and

$$\tilde{\beta}_\epsilon := \left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2} C_2\right) y^2 (1-y)^2.$$

Remark 4.1.2. *Without the quadratic term in our approximation we would have* $\tilde{\beta}_\epsilon(y) = \left(\frac{\sigma^2}{2}\right) y^2 \left(1-y\right)^2$, *and if* $\sigma_1 = \sigma_2 = 0$ *then* $\tilde{\beta}_\epsilon(y) \equiv 0$. *Hence, the quadratic term is included to ensure that* $\tilde{\beta}_\epsilon(y) > 0$ *for all* $y \in (0,1)$.

Remark 4.1.3. *Since we are approximating the exiting time of our stochastic replicator equation, we will define the "new" process as* $\hat{\mathbf{s}}(t) = \left(\hat{s}_1(t), \hat{s}_2(t)\right)$.

We are now ready to state the main theorem for this section. The statement of the theorem is given from the perspective of the dynamics of $\hat{s}_1(t)$, however, since $\hat{s}_2(t) = 1 - \hat{s}_t(t)$, understanding how $\hat{s}_1(t)$ evolves tells us how $\hat{s}_2(t)$ evolves as well.

Theorem 4.1.1. *Take* $\hat{\mathbf{s}}(t) = \left(\hat{s}_1(t), \hat{s}_2(t)\right)$ *in the remark above,* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, σ_i^2 *the variance of the* i^{th} *subpopulation,* C_1 *and* C_2 *defined above, and* $\mathbf{y}_0 = (y_0, y_0') \in \Delta_2$.

(i) If $a_{11} - a_{21} < \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \frac{\epsilon^2}{2} C_2$ and $a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) - \frac{\epsilon^2}{2} C_2$, *then* $P_{\mathbf{y}_0}\left(\lim_{t\to\infty} \hat{s}_1(t) = 0\right) = 1$.

(ii) If $a_{11} - a_{21} < \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \frac{\epsilon^2}{2} C_2$ and $a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) - \frac{\epsilon^2}{2} C_2$, *then* $P_{\mathbf{y}_0}\left(\limsup_{t\to\infty} \hat{s}_1(t) = 1\right) = P_{\mathbf{y}_0}\left(\liminf_{t\to\infty} \hat{s}_1(t) = 0\right) = 1$.

(iii) If $a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \frac{\epsilon^2}{2} C_2$ and $a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) - \frac{\epsilon^2}{2} C_2$, *then* $P_{\mathbf{y}_0}\left(\lim_{t\to\infty} \hat{s}_1(t) = 1\right) = 1$.

(iv) If $a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \frac{\epsilon^2}{2} C_2$ and $a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) - \frac{\epsilon^2}{2} C_2$, *then* $P_{\mathbf{y}_0}\left(\lim_{t\to\infty} \hat{s}_1(t) = 0\right) = \frac{f(1) - f(y_0)}{f(1) - f(0)}$ *and* $P_{\mathbf{y}_0}\left(\lim_{t\to\infty} \hat{s}_1(t) = 1\right) = \frac{f(y_0) - f(0)}{f(1) - f(0)}$.

Proof. We will prove this theorem by proving four lemmas, where the lemmas correspond to each particular case of inequalities. $\square$

Before we prove each lemma we give some intuition behind the inequalities in the theorem above, and use these details as a basis of our proofs. Using an integrating factor we have

$$f'(y) = k_1 \exp\left\{ -\int_{y_0}^{y} \frac{\tilde{\alpha}_\epsilon(z)}{\tilde{\beta}_\epsilon(z)} dz \right\},$$

for some constant k_1 and $y_0 \in (y_1, y_2)$. We will now work to simplify the integral in the exponent. Define $N_\epsilon = a_{12} - a_{22} + \sigma_2^2 + \epsilon(C_1 - \nu(B))+$ and $M_\epsilon = (a_{11} - a_{21} - \sigma_1^2) + (a_{22} - a_{12} - \sigma_2^2) - \epsilon^2 C_2$. Hence

$$\begin{aligned}
-\int_{y_0}^{y} \frac{\tilde{\alpha}_\epsilon(z)}{\tilde{\beta}_\epsilon(z)} dz &= -\left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} \int_{y_0}^{y} \frac{N_\epsilon + M_\epsilon z}{z(1-z)} dz \\
&= -\left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} \int_{y_0}^{y} \frac{N_\epsilon + M_\epsilon}{(1-z)} + \frac{N_\epsilon}{z} dz \\
&= \log\left((y/y_0)^{-\left(\frac{\sigma^2}{2}+\frac{\epsilon^2}{2}C_2\right)^{-1} N_\epsilon} ((1-y)/(1-y_0))^{\left(\frac{\sigma^2}{2}+\frac{\epsilon^2}{2}C_2\right)^{-1}(N_\epsilon + M_\epsilon)} \right).
\end{aligned} \tag{4.2}$$

So $f(y) = k_1 \displaystyle\int_{y_0}^{y} (z/y_0)^{-\left(\frac{\sigma^2}{2}+\frac{\epsilon^2}{2}C_2\right)^{-1} N_\epsilon} ((1-z)/(1-y_0))^{\left(\frac{\sigma^2}{2}+\frac{\epsilon^2}{2}C_2\right)^{-1}(N_\epsilon + M_\epsilon)} dz - k_2$, for some constant k_2.

Defining $u(y) = \dfrac{f(y) - f(y_1)}{f(y_2) - f(y_1)}$, $f(y) = f(y_1)$ for $y \in [0, y_1]$, and $f(y) = f(y_2)$ for $y \in [y_2, 1]$, we see that $\tilde{\alpha}_\epsilon(y)u'(y) + \tilde{\beta}(y)u''(y) = 0$ for $y \in (y_1, y_2)$, $u(y) = 0$ for $y \in [0, y_1]$, and $u(y) = 1$ for $y \in [y_2, 1]$.

Note that

$$\begin{aligned}
-\left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} N_\epsilon > -1 &\iff a_{22} - a_{12} - \sigma_2^2 - \epsilon(C_1 - \nu(B)) > -\sigma^2/2 - \frac{\epsilon^2}{2}C_2 \\
&\iff a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(B)) - \frac{\epsilon^2}{2}C_2
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} (N_\epsilon + M_\epsilon) > -1 &\iff a_{11} - a_{21} - \sigma_1^2 + \epsilon(C_1 - \nu(B)) - \epsilon^2 C_2 > -\sigma^2/2 - \frac{\epsilon^2}{2}C_2 \\
&\iff a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(B)) + \frac{\epsilon^2}{2}C_2.
\end{aligned}$$

We are also able to conclude that

$$-\left(\frac{\sigma^2}{2} + \frac{\epsilon^2}{2}C_2\right)^{-1} N_\epsilon < -1 \iff a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(B)) - \frac{\epsilon^2}{2}C_2$$

and

$$\left(\frac{\sigma^2}{2}+\frac{\epsilon^2}{2}C_2\right)^{-1}(N_\epsilon+M_\epsilon)<-1 \iff a_{11}-a_{21}<\frac{\sigma_1^2-\sigma_2^2}{2}-\epsilon(C_1-\nu(B))+\frac{\epsilon^2}{2}C_2.$$

Using arguments similar to those in Gihman and Skorohod ([8] **Part I** Chapter 4 §16) we have the following lemmas. We will only prove the first lemma, keeping in mind that the rest of the lemmas are proved similarly.

Lemma 4.1.1. *If we have the inequalities*

$$a_{11}-a_{21}<\frac{\sigma_1^2-\sigma_2^2}{2}-\epsilon(C_1-\nu(B))+\frac{\epsilon^2}{2}C_2 \text{ and } a_{22}-a_{12}>\frac{\sigma_2^2-\sigma_1^2}{2}+\epsilon(C_1-\nu(B))-\frac{\epsilon^2}{2}C_2,$$

$\left(f(x)\to\infty \text{ as } x\to 1 \text{ and } \lim\limits_{x\to 0} f(x)<\infty\right)$, *then*

$$P_{y_0}\left(\sup_{t>0}\hat{s}(t)<1\right)=P_{y_0}\left(\inf_{t>0}\hat{s}(t)=0\right)=P_{y_0}\left(\lim_{t\to\infty}\hat{s}(t)=0\right)=1.$$

Proof. We will follow the proof in [8]. Since $\lim\limits_{y_1\to 0}P_{y_0}\left(\hat{s}(t_{y_1y_2}(y_0))\geq y_2\right)=P_{y_0}\left(\sup\limits_{t>0}\hat{s}(t)\geq y_2\right)$, we have the equality $P_{y_0}\left(\sup\limits_{t>0}\hat{s}(t)\geq y_2\right)=\dfrac{f(y_0)-f(0)}{f(y_2)-f(0)}$. Letting $y_2\to 1$, we conclude that $P_{y_0}\left(\sup\hat{s}(t)<1\right)=1$. Also $P_{y_0}\left(\inf\limits_{t>0}\hat{s}(t)\leq y_1\right)\geq P_{y_0}\left(\hat{s}(t_{y_1y_2}(y_0))\leq y_1\right)=\dfrac{f(y_2)-f(y_0)}{f(y_2)-f(y_1)}\to 1$ as $y_2\to 1$.

Showing that $P_{y_0}\left(\limsup\limits_{t\to\infty}s(t)\geq y_2\right)=0$ will finish the proof. Define τ_q to be the first time passage of the process past or to the point $q<y_0$. By above we have $P_{y_0}(\tau_q<\infty)=1$. Furthermore, since τ_q is a stopping time $P_{y_0}\left(\sup\limits_{t>0}\hat{s}(t+\tau_q)\geq y_2\right)=P_{y_0}\left(\sup\limits_{t>0}\hat{s}(t)\geq y_2\middle|\hat{s}(0)=q\right)=\dfrac{f(q)-f(0)}{f(y_2)-f(0)}$. Moreover $P_{y_0}\left(\sup\limits_{t>0}\hat{s}(t+\tau_q)\geq y_2\right)=P_{y_0}\left(\sup\limits_{t>\tau_q}\hat{s}(t)\geq y_2\right)\geq P_{y_0}\left(\liminf\limits_{t\to\infty}\hat{s}(t)\geq y_2\right)$. Thus, taking $q\to 0$ we obtain $P_{y_0}\left(\liminf\limits_{t\to\infty}\hat{s}(t)\geq y_2\right)=0$. Therefore $\liminf\limits_{t\to\infty}\hat{s}(t)=0$ a.s. and we are done. ∎

Lemma 4.1.2. *If we have the inequalities*

$$a_{11}-a_{21}<\frac{\sigma_1^2-\sigma_2^2}{2}-\epsilon(C_1-\nu(B))+\frac{\epsilon^2}{2}C_2 \text{ and } a_{22}-a_{12}<\frac{\sigma_2^2-\sigma_1^2}{2}+\epsilon(C_1-\nu(B))-\frac{\epsilon^2}{2}C_2,$$

$(f(x)\to\infty \text{ as } x\to 1 \text{ and } f(x)\to\infty \text{ as } x\to 0)$, *then*

$$P_{y_0}\left(\lim\sup_{t\to\infty}\hat{s}(t)=1\right)=P_{y_0}\left(\lim\inf_{t\to\infty}\hat{s}(t)=0\right)=1.$$

Lemma 4.1.3. *If we have the inequalities*

$$a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(B)) + \frac{\epsilon^2}{2}C_2 \textit{ and } a_{22} - a_{12} < \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(B)) - \frac{\epsilon^2}{2}C_2,$$

$\Big(f(x) \to \infty$ *as* $x \to 0$ *and* $\lim_{x\to 1} f(x) < \infty\Big)$, *then*

$$P_{y_0}\left(\inf_{t>0} \hat{s}(t) > 0\right) = P_{y_0}\left(\sup_{t>0} \hat{s}(t) = 1\right) = P_{y_0}\left(\lim_{t\to\infty} \hat{s}(t) = 1\right) = 1.$$

Lemma 4.1.4. *If we have the inequalities*

$$a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(B)) + \frac{\epsilon^2}{2}C_2 \textit{ and } a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(B)) - \frac{\epsilon^2}{2}C_2,$$

$(f(x) < \infty\ \forall x)$, *then*

$$P_{y_0}\left(\sup_{t>0} \hat{s}(t) < 1\right) = P_{y_0}\left(\lim_{t\to\infty} \hat{s}(t) = 0\right) = \frac{f(1) - f(y_0)}{f(1) - f(0)}$$

and

$$P_{y_0}\left(\inf_{t>0} \hat{s}(t) > 0\right) = P_{y_0}\left(\lim_{t\to\infty} \hat{s}(t) = 1\right) = \frac{f(y_0) - f(0)}{f(1) - f(0)}.$$

This is very similar to the results found by Fudenberg and Harris [7], with the added or subtracted piece $\epsilon(C_1 - \nu(B)) - \frac{\epsilon^2}{2}C_2$. Recall that $\nu(B)$ gives us the intensity of the jumps. If we have that $C_1 > \nu(B)$ then $h_2(x) < 0$ for a significant amount of $x \in B$, and if $C_1 < \nu(B)$ then $h_2(x) > 0$ for a significant amount of $x \in B$. Furthermore, when $h_2(x) > 0$ for a significant amount of $x \in B$, $\frac{\epsilon^2}{2}C_2$ is a very small, and when $h_2(x) < 0$ for a significant amount of $x \in B$, $\frac{\epsilon^2}{2}C_2$ is a large. This term would give a very little or considerable impact to the inequalities, respectively.

To show the importance of this term we will analyze the example of $\epsilon > 0$, $h_2(x) < 0$ for all $x \in B$ and $\epsilon(C_1 - \nu(B)) - \frac{\epsilon^2}{2}C_2 > 0$. This would make it more difficult for the difference $a_{22} - a_{12}$ to satisfy the inequality $a_{22} - a_{12} > \frac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon(C_1 - \nu(B)) - \frac{\epsilon^2}{2}C_2$ and easier for $a_{11} - a_{21} > \frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon(C_1 - \nu(B)) + \frac{\epsilon^2}{2}C_2$. Hence, it is a bit more likely that $\hat{s}(t) \to 1$ as $t \to \infty$ a.s. Since $h_1(t) = h_2(t) + \epsilon$, ($1^{st}$ subpopulation jump dominance), the population playing the first strategy is not affected negatively as much as the population playing the second strategy, and thus this result is what we would expect.

Remark 4.1.4. *In most applications to this type of modeling, the Poisson measure and not the compensated Poisson measure is used. This is a simple adjustment applied to our method. The replacement of the term* $\epsilon(C_1 - \nu(B))$ *with* ϵC_1 *will suffice. It should also be noted that this new term has much more of an impact to the inequalities of the payoffs.*

4.2 Simulations of the True and Approximated Stochastic Replicator Dynamic

Since the results in the previous section are given by an approximation, we need to get an understanding of how close this approximation is to the actual process. To accomplish this task, we will simulate each of the processes.

Note 4.2.1. *For the approximation, $\hat{s}(t)$, we have the infinitesimal generator,*

$$\begin{aligned} Lf(y) &= y(1-y)\left[\left(a_{12}-a_{22}+\sigma_2^2+\epsilon(C_1-\nu(B))\right)+\{(a_{11}-a_{21}-\sigma_1^2)+(a_{22}-a_{12}-\sigma_2^2)-\epsilon^2C_2\}y\right]f'(y)\\ &+\left(\frac{\sigma^2}{2}+\frac{\epsilon^2}{2}C_2\right)y^2(1-y)^2f''(y)\\ &=\tilde{\alpha}(y)f'(y)+\tilde{\beta}(y)f''(y), \end{aligned}$$

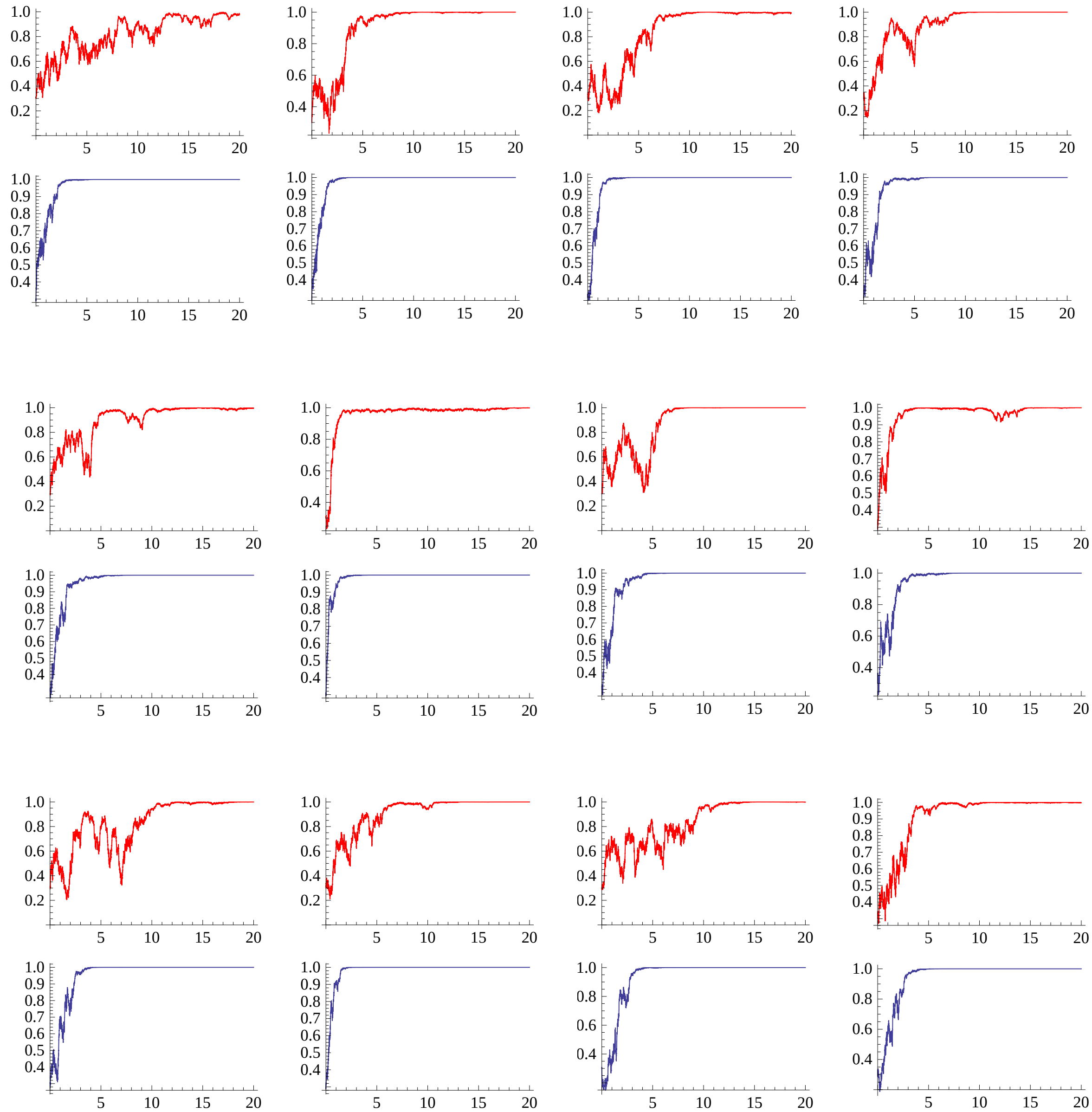
and considering Kolmogorov's backward equation, we conclude that the process for the approximation is of the form

$$d\hat{s}(t)=\tilde{\alpha}\big(\hat{s}(t)\big)dt+\sqrt{2\tilde{\beta}\big(\hat{s}(t)\big)}dW(t)$$

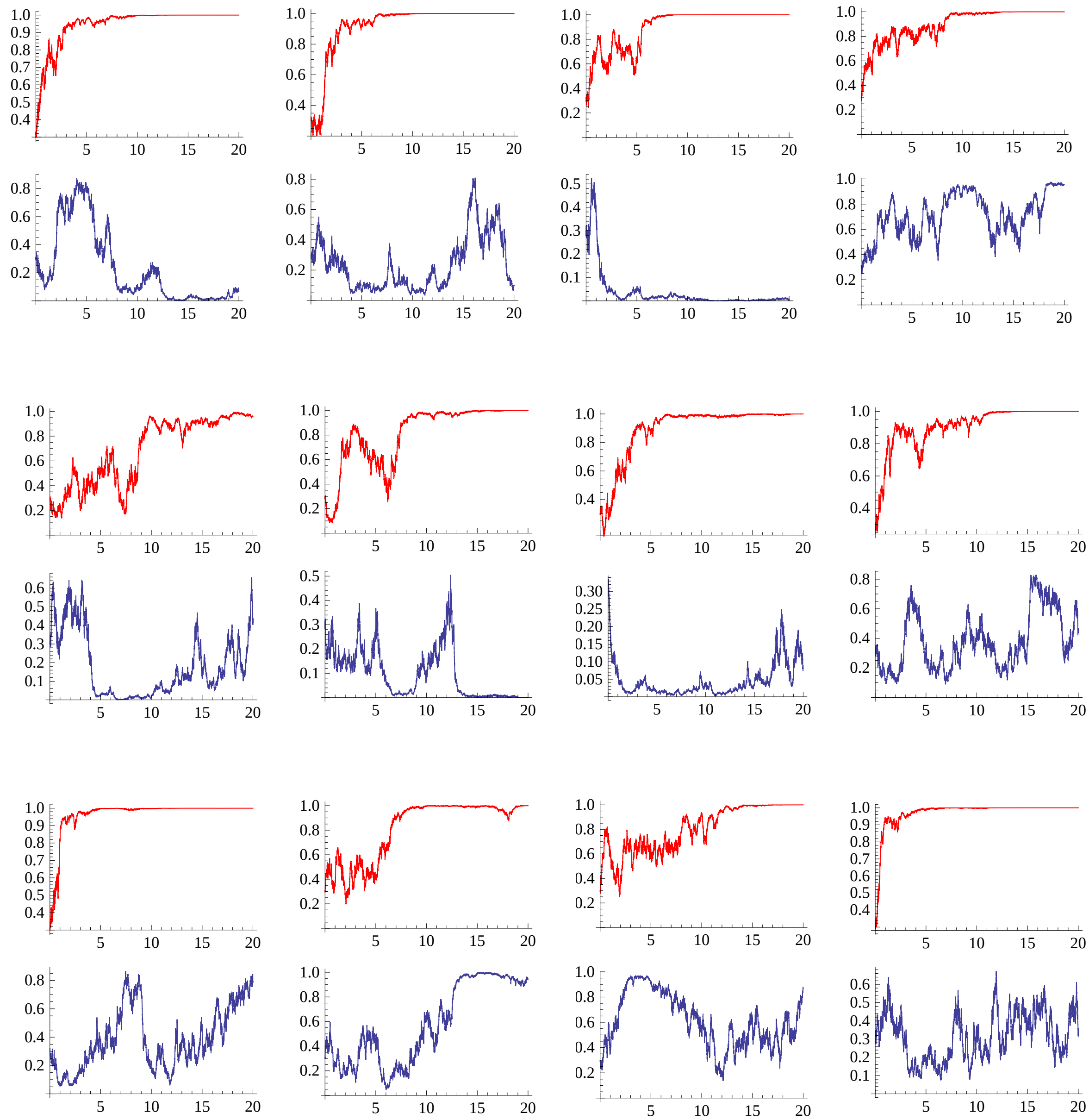
where $W(t)$ is the Brownian motion given in §2.2.

Throughout this section, the red sample paths will be the approximated processes, while the blue sample paths represents the true replicator process. In the simulations we will see how the size of ϵ, (which was defined to be h_1-h_2, can make a tremendous difference in behavior of the processes), as well the size of $\nu(\mathbb{R})$ (the intensity of the Poisson measure). Under small Poisson perturbations, the long run behavior of both processes are essentially equivalent.

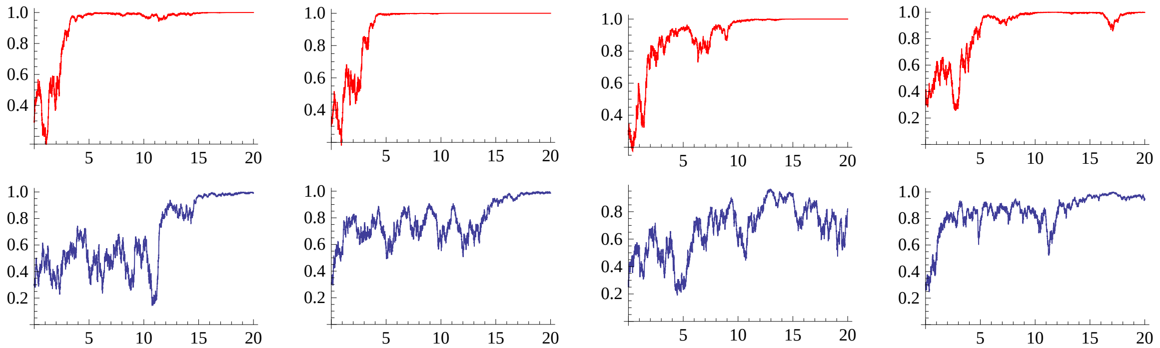
Example 4.2.1. *Take $a_{11}=1.2$, $a_{21}=1$, $a_{22}=0$, $a_{12}=1$, which is a strategy one dominate game, in which over time the entire population would be playing strategy one. If we perturb this game by taking $\sigma_1^2=.2$, $\sigma_2^2=.8$, $\nu\big(\mathbb{R}\big)=.2$, $h_1=.5$ and $h_2=.3$, then according to Theorem 4.1.1, the entire population still would eventually be playing strategy one. Simulating these processes yields exactly that:*

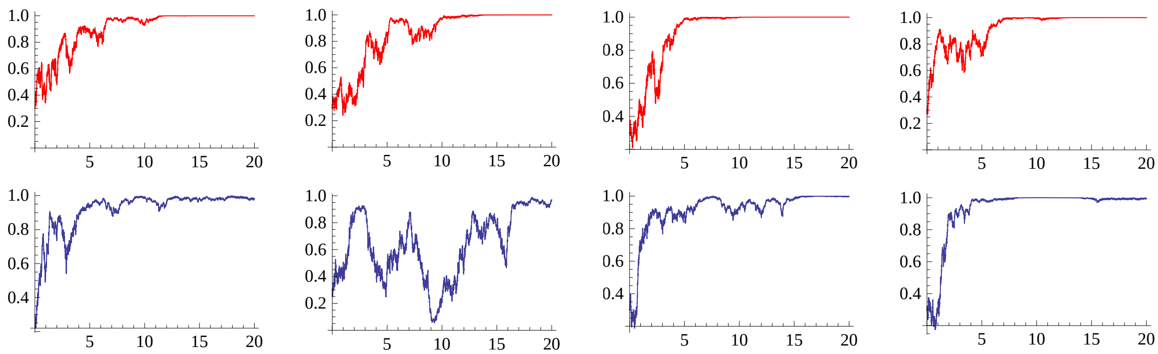


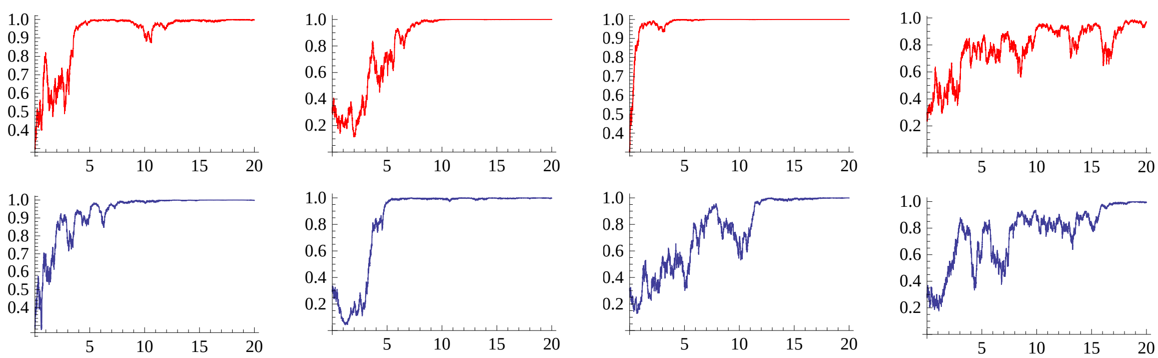
Switching the values of h_1 *and* h_2 *to* $h_1 = .3$ *and* $h_2 = .5$*, the jumps now favor the* 2^{nd} *subpopulation instead of the* 1^{st} *subpopulation. For the approximated process, the behavior is the same, since this change did not affect the inequalities of Theorem 4.1.1, however, the true process takes on more of a recurrent characteristic:*



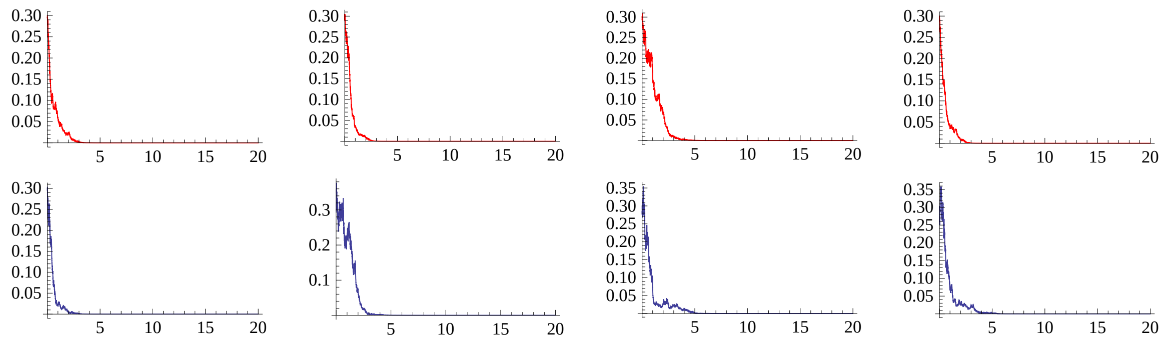
This tells us that $h_1 - h_2$ may not be an appropriate size for ϵ. Increasing the size of h_1 to .43 yields the same long run behavior:

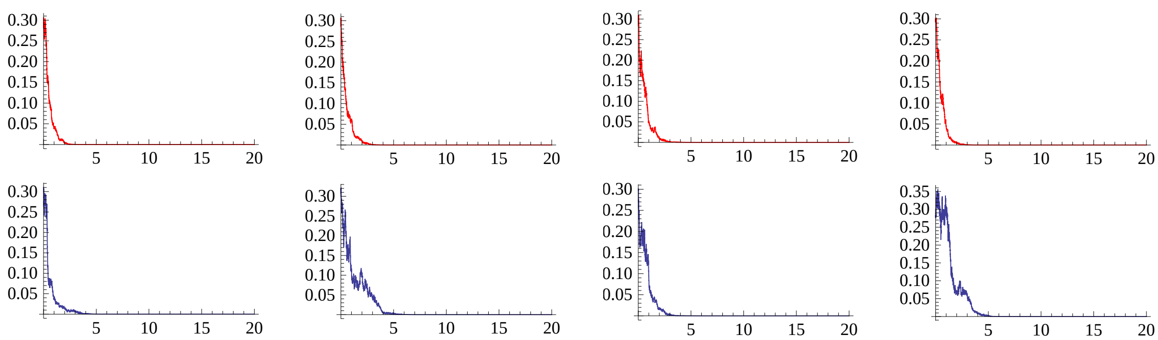


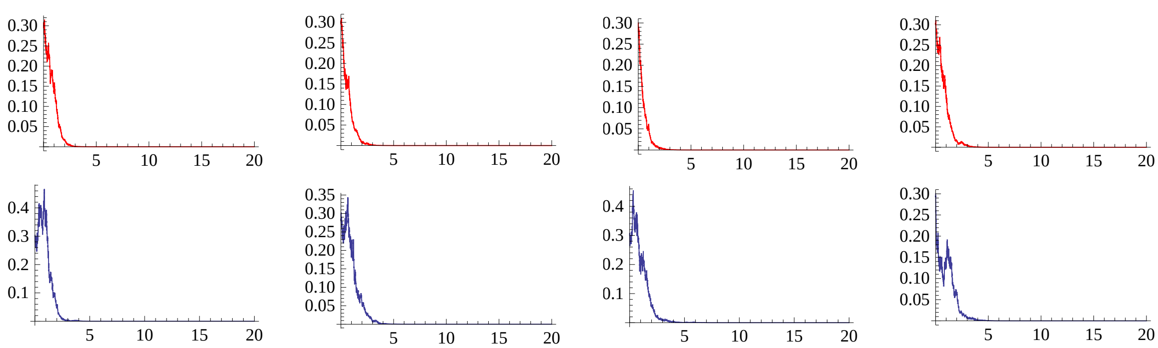




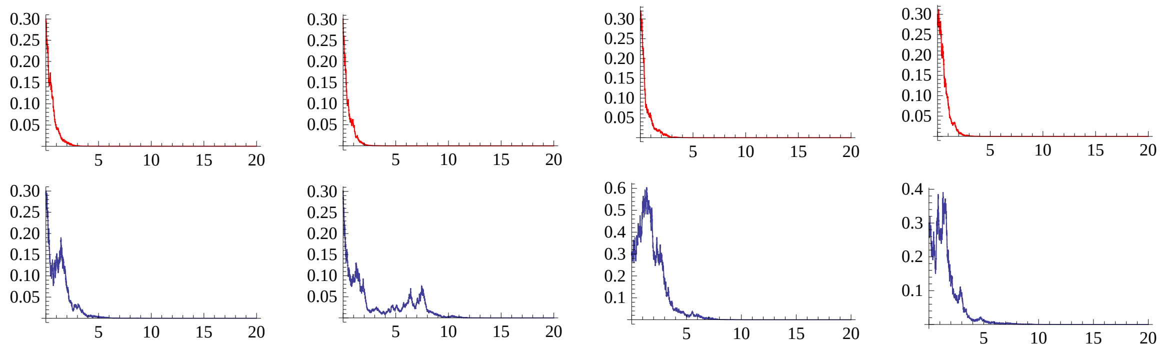
Example 4.2.2. *Now, take $a_{11} = 1$, $a_{21} = 1$, $a_{22} = 3$, $a_{12} = 1$, which is a strategy two dominate game, and hence in the long run the entire population will play strategy two. If we perturb this game by taking $\sigma_1^2 = .4$, $\sigma_2^2 = .1$, $\nu(\mathbb{R}) = .3$, $h_1 = .7$ and $h_2 = .6$, the Theorem 4.1.1 tells us that the entire population will eventually be playing strategy two. After simulation, we see agreement with both processes:*

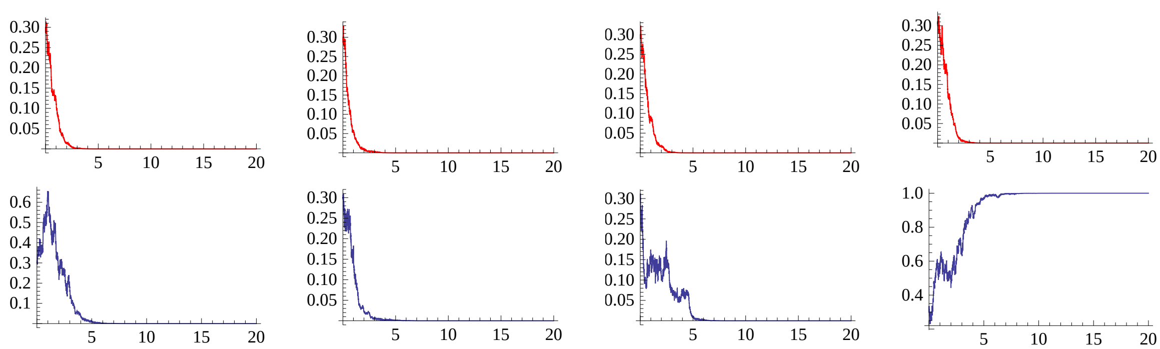


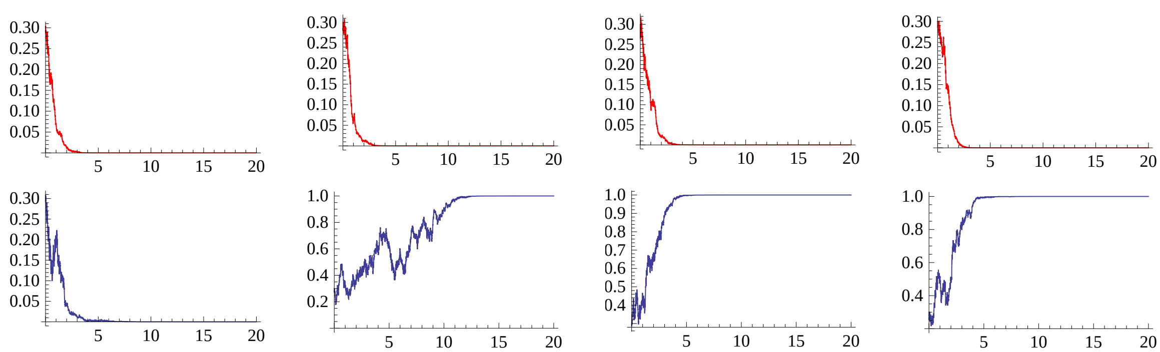




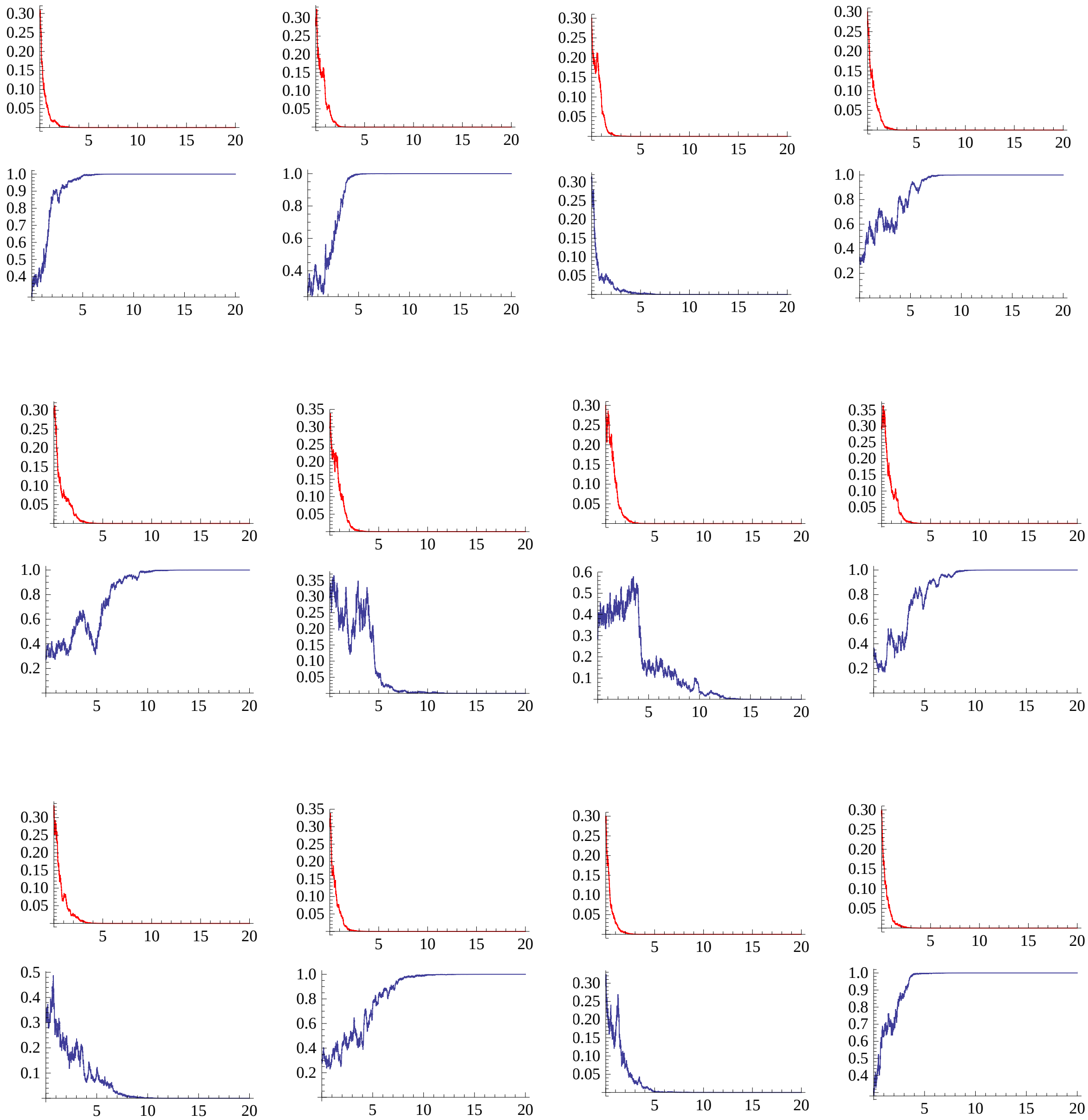
Although the jumps favor the 1^{st} subpopulation and ϵ is larger than in the previous example, the difference of $a_{22} - a_{21}$ is large. Switching the values of h_2 to $h_2 = .5$ in order to slightly increases the size of ϵ, the behavior does not change for the approximated process, but the true process now converges to both 0 and 1:





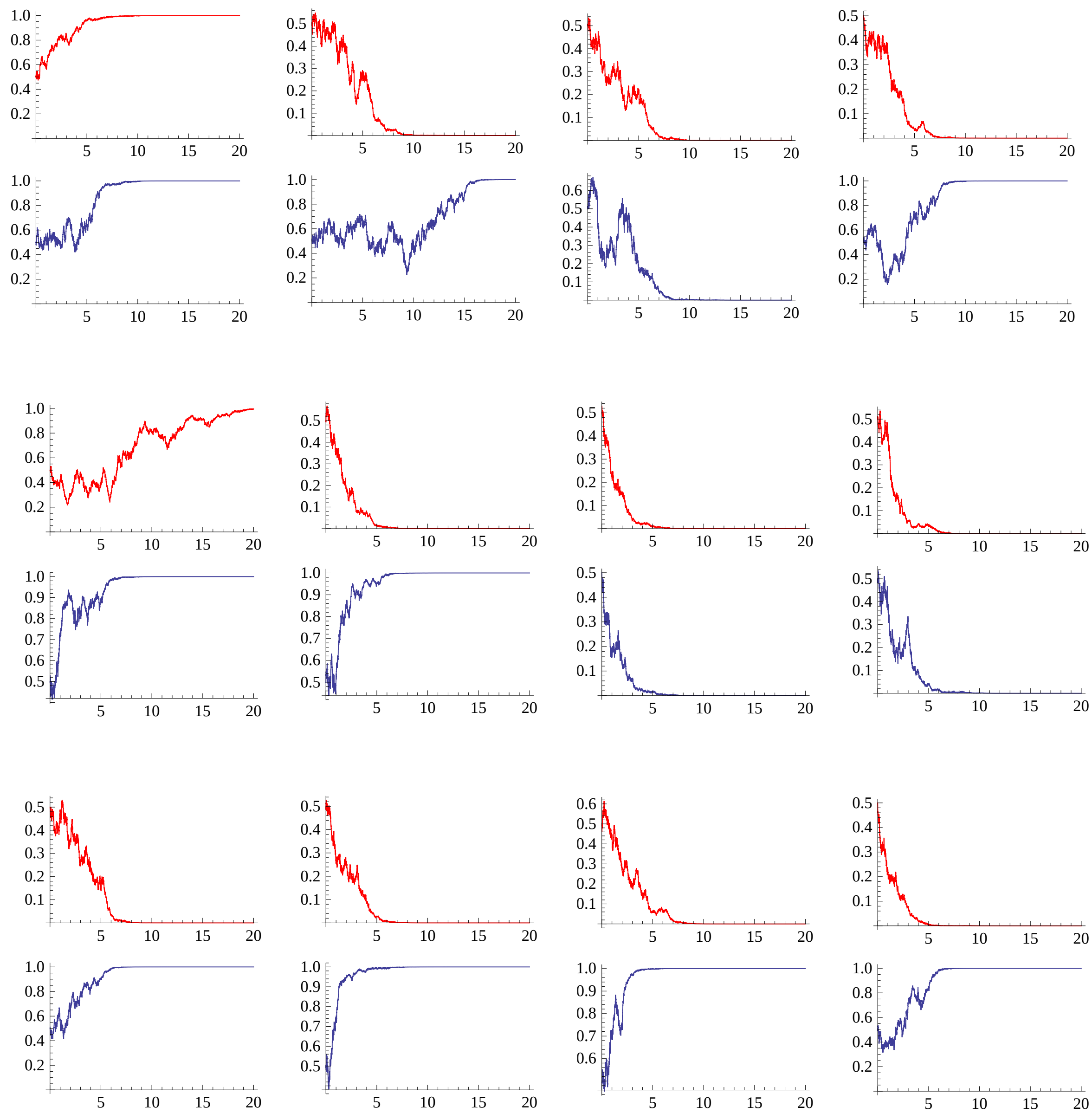


Even though $a_{22} - a_{21}$ is still large, a slight increase in the size of ϵ gives quite different behavior. Finally, if we increase the value of $\nu(\mathbb{R})$ to 2, with the original values of the jumps, $h_1 = .7$ and $h_2 = .6$, we see similar behavior of the true replicator process when $h_2 = .5$:

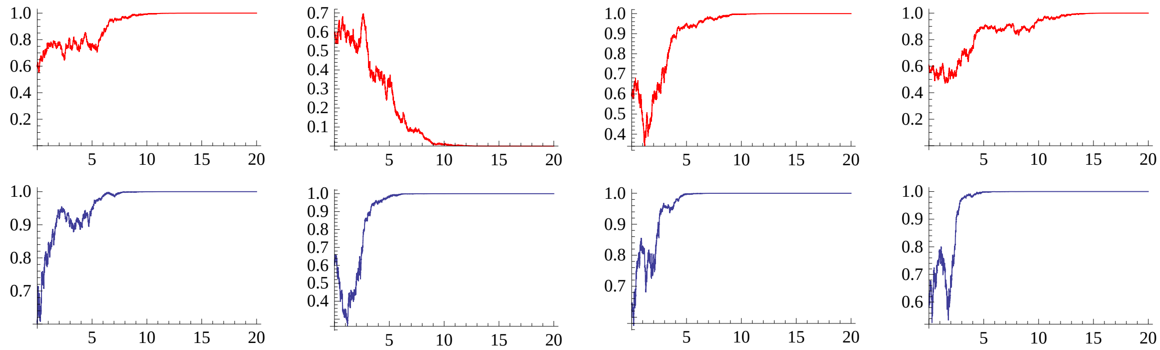


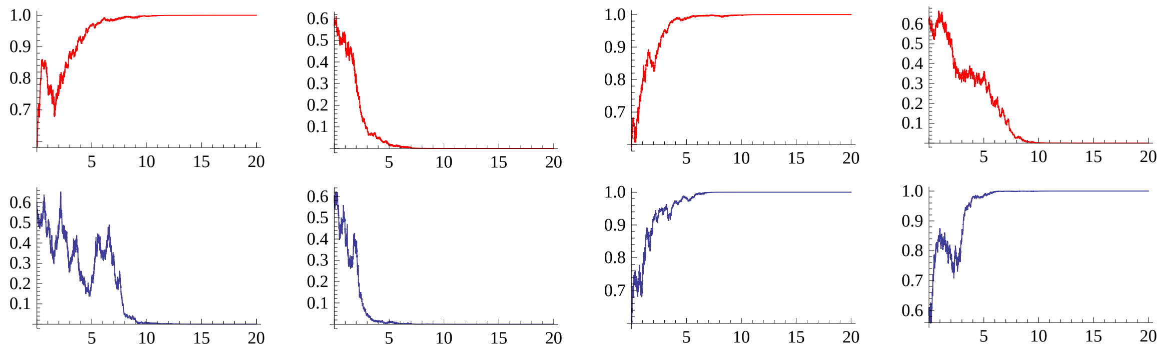
Considering how increasing the value of ϵ changed behavior, this is what we would expect. Thus, we need the term $\int_{\mathbb{R}} (h_1 - h_2)\nu(dx)$ *to be generally small.*

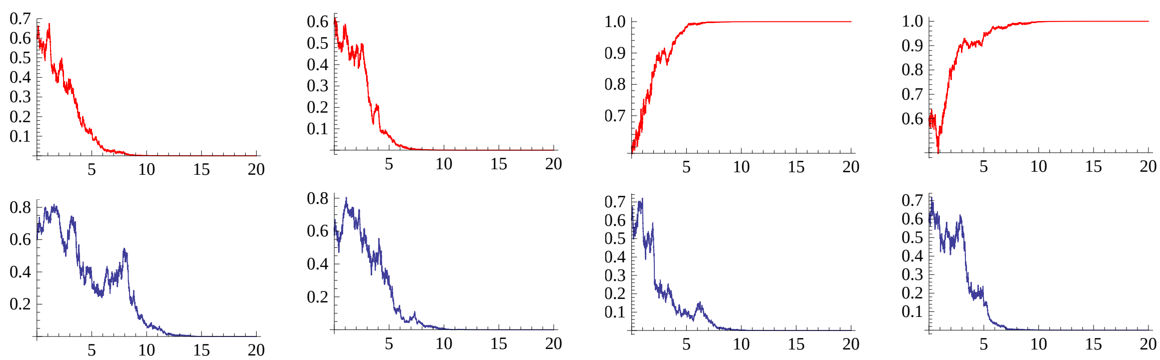
Example 4.2.3. *For this example, we will simulate a coordination game. Take* $a_{11} = 1$*,* $a_{21} = 0$*,* $a_{22} = 1$*,* $a_{12} = 0$*, and the perturbations as* $\sigma_1^2 = .4$*,* $\sigma_2^2 = .1$*,* $\nu(\mathbb{R}) = .3$*,* $h_1 = .7$ *and* $h_2 = .6$*. Since the perturbations are the same as the previous example, we know that the size of the jumps and the intensity of the Poisson measure are appropriate. Theorem 4.1.1 tells us that the approximate process will converge to both 0 and 1 with the respective probabilities. The simulation gives exactly that for each process:*



however, there appears to be different probabilities for convergence to each point. Decreasing the value of ϵ *to .05 (by setting* $h_2 = .65$*) yields the same number of times each process converges to each point:*







which tells us that the probabilities to converge to each point are very close (with respect of the initial value at .6).

4.3 Error Approximation

Consider the equation

$$\alpha(y)u'(y) + \beta(y)u''(y) + \int_{\mathbb{R}} \Big[u\Big(y + \gamma(y,x)\Big) - u(y) \Big]\, \nu(dx) = 0, \tag{4.3}$$

where we recall that $\alpha(y) = y(1-y)\big(-b + (a+b)y\big)$, $a := a_{11} - a_{21} - \sigma_1^2 + \int_{\mathbb{R}} \Big(h_1(x) - h_2(x)\Big)\nu(dx)$ and $b := a_{22} - a_{12} - \sigma_2^2 - \int_{\mathbb{R}} \Big(h_1(x) - h_2(x)\Big)\nu(dx)$, $\beta(y) = \dfrac{\sigma^2}{2} y^2(1-y)^2$, and $y + \gamma(y,x) = \dfrac{y\big[1 + h_1(x)\big]}{y\big[h_1(x) - h_2(x)\big] + 1 + h_2(x)}$. We want to consider this equation on a closed interval $[y_1, y_2]$ that is a subset of $(0,1)$, where $0 < y_1 < y_2 < 1$, and consider the space F of continuous functions on $[y_1, 1]$ defined by:

$$F = \Big\{ u \in C\big([y_1, 1]\big) \Big| u(y_1) = 0, u(y) = 1 \text{ for all } y_2 \le y \le 1 \Big\}.$$

Although Tuckwell [23] tells us that a solution for (4.3) exists, we consider a different method to show that this integro-differential equation supports such a solution. We certainly don't need to consider the equation for $y < y_1$. We should be able to arrange that $u(y_1) = 0$ by the choice of a constant of integration. If we restrict attention to functions $h_i(x)$ satisfying

$$h_1(x) > h_2(x) > h > -1$$

for some h, then

$$\begin{aligned}
\gamma(y, x) &= \frac{y\big[1 + h_1(x)\big]}{y\big[h_1(x) - h_2(x)\big] + 1 + h_2(x)} - y \\
&= \frac{y\big[1 + h_1(x)\big] - y^2\big[h_1(x) - h_2(x)\big] - y\big[1 + h_2(x)\big]}{y\big[h_1(x) - h_2(x)\big] + 1 + h_2(x)} \\
&= \frac{y(1 - y)\big[h_1(x) - h_2(x)\big]}{y\big[h_1(x) - h_2(x)\big] + 1 + h_2(x)} > 0.
\end{aligned}$$

This shows that the integral term in (4.3) is zero for all $y_2 \le y \le 1$, and the derivative terms are zero since the functions under consideration are constant in this range. Thus, we will be able to replace $\gamma(y, x)$ with $\gamma(y, x)\varphi(y)$, where φ is the characteristic function $1_{[y_1, y_2]}$. Our problem now is

$$\alpha(y)u'(y) + \beta(y)u''(y) + \int_{\mathbb{R}} \Big[u\big(y + \gamma(y, x)\varphi(y)\big) - u(y)\Big]\nu(dx) = 0, \tag{4.4}$$

for $u \in F$.

We observe that the difference in the integral term can be written as

$$u\Big(y + \gamma(y, x)\varphi(y)\Big) - u(y) = \int_y^{y+\gamma(y,x)\varphi(y)} u'(z)dz,$$

and so

$$\alpha(y)u'(y) + \beta(y)u''(y) = -\int_{\mathbb{R}} \int_y^{y+\gamma(y,x)\varphi(y)} u'(z)dz\nu(dx).$$

Multiplying by an appropriate integrating factor and integrating gives us

$$\Big(f_{y_1}(u)u'(y)\Big)' = -f_{y_1}(y)\frac{1}{\beta(y)} \int_{\mathbb{R}} \int_y^{y+\gamma(y,x)\varphi(y)} u'(z)dz\nu(dx),$$

where

$$f_{y_1}(y) = \exp\left\{\int_{y_1}^{y} \frac{\alpha(w)}{\beta(w)} dw\right\} = \exp\left\{\frac{2}{\sigma^2}\int_{y_1}^{y} \frac{-b+(a+b)w}{w(1-w)} dw\right\} = \left(\frac{y}{y_1}\right)^{-2b/\sigma^2}\left(\frac{1-y}{1-y_1}\right)^{-2a/\sigma^2}.$$

We now make a change of variables from the function u to the function $v(y) = f_{y_1}(y)u'(y)$. We are able to recover an appropriate function u since

$$u(y) = \int_{y_1}^{y} \frac{v(s)}{f_{y_1}(s)} ds,$$

and hence $u(y_1) = 0$ as required by F. Our simplified equation is now of the form

$$v'(y) = -f_{y_1}(y)\frac{1}{\beta(y)}\int_{\mathbb{R}}\int_{y}^{y+\gamma(y,x)\varphi(y)} \frac{v(z)}{f_{y_1}(z)} dz\nu(dx).$$

Notice that

$$\frac{f_{y_1}(y)}{f_{y_1}(z)} = \frac{\exp\left\{\int_{y_1}^{y} \frac{\alpha(w)}{\beta(w)} dw\right\}}{\exp\left\{\int_{y_1}^{z} \frac{\alpha(w)}{\beta(w)} dw\right\}} = \exp\left\{\int_{z}^{y} \frac{\alpha(w)}{\beta(w)} dw\right\} = f_z(y).$$

Using this and integrating once gives us

$$v(y) = -\int_{y_1}^{y} \frac{1}{\beta(s)}\int_{\mathbb{R}}\int_{s}^{s+\gamma(s,x)\varphi(s)} f_z(s)v(z)dz\nu(dx)ds + C. \tag{4.5}$$

Without loss of generality, we consider the explicit problem

$$v(y) = -\int_{y_1}^{y} \frac{1}{\beta(s)}\int_{\mathbb{R}}\int_{s}^{s+\gamma(s,x)\varphi(s)} f_z(s)v(z)dz\nu(dx)ds + 1, \tag{4.6}$$

since multiplying this by C will give us a solution of (4.5).

Remark 4.3.1. *For our ultimate u to lie in F, we need*

$$u(y_2) = \int_{y_1}^{y_2} \frac{v(s)}{f_{y_1}(s)} ds = 1,$$

and this can be achieved by an appropriate choice of C.

It comes down, then, to finding a solution of (4.6) in $C([y_1, y_2])$. We consider this in the context of constant small jump differences. Therefore we take a small $\epsilon \in \mathbb{R}$, $\epsilon > 0$ and set $h_1(x) = h_2(x) + \epsilon$. Then a is replaced by $a_\epsilon = a + \epsilon\nu(\mathbb{R})$ and b with $b_\epsilon = b - \epsilon\nu(\mathbb{R})$. Furthermore, $\gamma(y, x)$ is replaced with

$$\gamma_\epsilon(y, x) = \frac{\epsilon(1-y)}{\epsilon y + 1 + h_2(x)}.$$

Let

$$T_\epsilon v(y) = -\int_{y_1}^{y} \frac{1}{\beta(s)} \int_{\mathbb{R}} \int_{s}^{s+\gamma_\epsilon(s,x)\varphi(s)} f_z(s,\epsilon) v(z) dz \nu(dx) ds,$$

for

$$f_z(s,\epsilon) = \left(\frac{s}{z}\right)^{-2b_\epsilon/\sigma^2} \left(\frac{1-s}{1-z}\right)^{-2a_\epsilon/\sigma^2}.$$

Then our problem for v is

$$(I - T_\epsilon)v = 1, \text{ i.e. } v = (I - T_\epsilon)^{-1}1, \tag{4.7}$$

provided that $I - T_\epsilon$ is invertible.

First we note that T_ϵ maps $C\big([y_1, y_2]\big)$ to itself. The characteristic function φ guarantees that T_ϵ is a well define map on $C\big([y_1, y_2]\big)$. Since T_ϵ involves no more than integrations involving continuous functions bounded on $[y_1, y_2]$, the value $T_\epsilon v$ is again a function in $C([y_1, y_2])$. Finally,

$$\begin{aligned}
\Big|T_\epsilon v(y)\Big| &\le \int_{y_1}^{y_2} \frac{1}{\beta(s)} \int_{\mathbb{R}} \int_{s}^{s+\gamma_\epsilon(s,x)\varphi(s)} f_z(s,\epsilon) |v(z)| dz \nu(dx) ds \\
&\le \int_{y_1}^{y_2} \frac{1}{\beta(s)} \int_{\mathbb{R}} \int_{s}^{s+\gamma_\epsilon(s,x)\varphi(s)} f_z(s,\epsilon) dz \nu(dx) ds \; ||v||_\infty \\
&\le \int_{y_1}^{y_2} \frac{1}{\beta(s)} \max_{z\in[y_1,y_2]} f_z(s,\epsilon) \int_{\mathbb{R}} \int_{s}^{s+\gamma_\epsilon(s,x)\varphi(s)} dz \nu(dx) ds \; ||v||_\infty \\
&= \int_{y_1}^{y_2} \frac{1}{\beta(s)} \max_{z\in[y_1,y_2]} f_z(s,\epsilon) \int_{\mathbb{R}} \gamma_\epsilon(s,x)\varphi(s) \nu(dx) ds \; ||v||_\infty \\
&= \epsilon \int_{y_1}^{y_2} \int_{\mathbb{R}} \frac{1}{\beta(s)} \max_{z\in[y_1,y_2]} f_z(s,\epsilon) \frac{(1-s)}{\epsilon s + 1 + h_2(x)} ds \; \nu(dx) \; ||v||_\infty \\
&\le \epsilon \nu(\mathbb{R}) \int_{y_1}^{y_2} \frac{1}{\beta(s)} \max_{z\in[y_1,y_2]} f_z(s,\epsilon) \frac{(1-s)}{\epsilon s + 1 + h} ds \; ||v||_\infty \\
&\le \epsilon \nu(\mathbb{R}) \max_{s\in[y_1,y_2]} \left[\frac{1}{\beta(s)} \max_{z\in[y_1,y_2]} f_z(s,\epsilon) \frac{(1-s)}{\epsilon s + 1 + h}\right] (y_2 - y_1) \; ||v||_\infty.
\end{aligned}$$

Therefore

$$||T_\epsilon v||_\infty = \max_{y\in[y_1,y_2]} |T_\epsilon v(y)| \le \epsilon K ||v||_\infty,$$

which implies that $||T_\epsilon|| \le \epsilon K$, where

$$K = \nu(\mathbb{R}) \max_{s\in[y_1,y_2]} \left[\frac{1}{\beta(s)} \max_{z\in[y_1,y_2]} f_z(s,\epsilon) \frac{(1-s)}{\epsilon s + 1 + h}\right] (y_2 - y_1) < \infty.$$

Since $||T_\epsilon|| < 1$ for ϵ sufficiently small, we conclude by Theorem 3.2.8 in Taira [21] that $I - T_\epsilon$ is invertible and we actually have the series representation of the solution

$$v = 1 + T_\epsilon 1 + T_\epsilon^2 1 + T_\epsilon^3 1 + \cdots$$

that is uniformly convergent, i.e., convergent in $C\big([y_1, y_2]\big)$, and where each successive term is $O(\epsilon^n)$ as $n \to \infty$.

Chapter 5

Characterizations of the General Model

5.1 Evolutionary Stable Strategies

Much of the work done in this chapter is based off of a paper written by Imhof [11]. Take $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ as the standard basis of $\mathbb{R}^n$. Define $\Delta_n = \left\{\mathbf{y} \in \mathbb{R}^n : y_i > 0 \text{ for all } i \text{ and } \sum y_i = 1\right\}$ (the open simplex) and $\overline{\Delta}_n$ as it's closure. Furthermore, for the Euclidean norm $|\cdot|$, define $U_\delta(\mathbf{y}_0) = \{\mathbf{y} \in \Delta_n : |\mathbf{y}_0 - \mathbf{y}| < \delta\}$ and $\tau_G = \inf\{t > 0 : \mathbf{s}(t) \in G\}$, where G is a Borel set. Recall that $s_i(t)$ is of the form

$$\begin{aligned} ds_i(t) = s_i(t-) &\Bigg[\big(As(t-)\big)_i - \sum_j s_j(t-)\big(As(t-)\big)_j + \sum_j s_j(t-)\sigma_j^2 - s_i(t-)\sigma_i^2 \\ &+ \int_B \left(\frac{1 + h_i(x)}{1 + \sum_j s_j(t-)h_j(x)} - 1 + \sum_j s_j(t-)h_j(x) - h_i(x)\right)\nu(dx)\Bigg]\, dt \\ &+ s_i(t-)\left(\sigma_i dW_i(t) - \sum_j s_j(t-)\sigma_j dW_j(t)\right) \\ &+ s_i(t-)\int_B \left(\frac{1 + h_i(x)}{1 + \sum_j s_j(t-)h_j(x)} - 1\right)\tilde{N}(dt, dx). \end{aligned} \tag{5.1}$$

Define $\mathbf{h}(\mathbf{x}) = (h_1(x), h_2(x), \ldots, h_n(x))^T$. With a little work one can see that

$$d\mathbf{s}(t) = D^1(\mathbf{s}(t-), \mathbf{h}(\mathbf{x}))dt + D^2(\mathbf{s}(t-))d\mathbf{W}(t) + \int_{\mathbb{R}} D^3(\mathbf{s}(t-), \mathbf{h}(\mathbf{x}))\tilde{N}(dt, dx)$$

where

$$\begin{aligned} D^1(\mathbf{y}, \mathbf{h}(\mathbf{x})) = {} & [\mathbf{diag}(y_1, \ldots, y_n) - \mathbf{y}\mathbf{y}^T][A - \mathbf{diag}(\sigma_1^2, \ldots, \sigma_n^2)]\mathbf{y} \\ &+ \int_{\mathbb{R}} \Bigg(\mathbf{y}\mathbf{h}(x)^T - \mathbf{diag}(h_1(x), \ldots, h_n(x)) \\ &+ \frac{1}{1 + \mathbf{y}^T\mathbf{h}(x)}\mathbf{diag}(1 + h_1(x), \ldots, 1 + h_n(x)) - \mathbf{diag}(1, \ldots, 1)\Bigg)\mathbf{y}\ \nu(dx), \end{aligned}$$

$$D^2(\mathbf{y}, \mathbf{h}(\mathbf{x})) = [\mathbf{diag}(y_1, \ldots, y_n) - \mathbf{y}\mathbf{y}^T]\mathbf{diag}(\sigma_1, \ldots, \sigma_n),$$

and

$$D^3(\mathbf{y},\mathbf{h}(\mathbf{x})) = \left(\frac{1}{1+\mathbf{y}^T\mathbf{h}(x)}\mathbf{diag}(1+h_1(x),\ldots,1+h_n(x)) - \mathbf{diag}(1,\ldots,1)\right)\mathbf{y},$$

Denote $\mathcal{A}_J$ as the second order integro-differential operator. For $f \in C^2(\Delta_n)$, we see that

$$\begin{aligned}\mathcal{A}_J f(\mathbf{y}) &= \sum_j D_j^1(\mathbf{y},\mathbf{h}(x))\frac{\partial f}{\partial y_j}(\mathbf{y}) + \frac{1}{2}\sum_{j,k}\gamma_{jk}(\mathbf{y})\frac{\partial^2 f}{\partial y_j \partial y_k}(\mathbf{y}) \\ &- \sum_j \int_{\mathbb{R}} D_j^3(\mathbf{y},\mathbf{h}(x))\nu(dx)\frac{\partial f}{\partial y_j}(\mathbf{y}) + \int_{\mathbb{R}}\Big(f\big(D^3(\mathbf{y},\mathbf{h}(\mathbf{x}))+\mathbf{y}\big) - f(\mathbf{y})\Big)\nu(dx),\end{aligned}$$

where D_j^i is the j^{th} coordinate of the function D^i and $\gamma_{jk}(\mathbf{y}) = \sum_l c_{jl}(\mathbf{y})c_{kl}(\mathbf{y})$ for $c_{jl}(\mathbf{y}) = \begin{cases} y_j(1-y_j)\sigma_j, & j = l \\ -y_j y_l \sigma_l & j \neq l \end{cases}$.
Notice that we can simplify the operator so that

$$\begin{aligned}\mathcal{A}_J f(\mathbf{y}) &= \sum_j \tilde{D}_j^1(\mathbf{y},\mathbf{h}(x))\frac{\partial f}{\partial y_j}(\mathbf{y}) + \frac{1}{2}\sum_{j,k}\gamma_{jk}(\mathbf{y})\frac{\partial^2 f}{\partial y_j \partial y_k}(\mathbf{y}) \\ &+ \int_{\mathbb{R}}\Big(f\big(D^3(\mathbf{y},\mathbf{h}(\mathbf{x}))+\mathbf{y}\big) - f(\mathbf{y})\Big)\nu(dx),\end{aligned}$$

for

$$\tilde{D}_i^1(\mathbf{y},\mathbf{h}(x)) = y_i(\mathbf{e}_i - \mathbf{y})^T[A - \mathbf{diag}(\sigma_1^2,\ldots,\sigma_n^2)]\mathbf{y} + \int_{\mathbb{R}} y_i\left(\sum_k y_k h_k(x) - h_i(x)\right)\nu(dx).$$

Before we prove a theorem about evolutionary stable strategies we will show that for some k our stochastic replicator equation becomes close to $\mathbf{e}_k$. Since the corner points of the simplex are absorbing, this is how we would expect the process to behave.

Theorem 5.1.1. *Take $\mathbf{s}(t)$ to be an n-dimensional stochastic replicator dynamics (5.1), the matrix A as an arbitrary payoff matrix, and for $\epsilon > 0$, $\tau_\epsilon := \inf\Big\{t > 0 : s_k(t) \geq 1-\epsilon \text{ for some } k \in \{1,2,\ldots,n\}\Big\}$. Then for $\mathbf{y} \in \Delta_n$,*

$$\mathbb{E}_{\mathbf{y}}[\tau_\epsilon] < \infty,$$

and

$$P_{\mathbf{y}}\Big\{\sup_{t>0}\max\{s_1(t),\ldots,s_n(t)\} = 1\Big\} = 1.$$

Proof. We will follow the proof of Theorem 4.3 in Imhof [11]. For $\alpha > 0$ and $\mathbf{y} \in \overline{\Delta}_n$ define the positive function $g(\mathbf{y}) = ne^\alpha - \sum_k e^{\alpha y_k}$. Define the "new" payoff matrix $\tilde{A} := A - \mathbf{diag}(\sigma_1^2,\ldots,\sigma_n^2)$ and the infinitesimal generator

$\mathcal{A}_J$. Then

$$\mathcal{A}_J g(\mathbf{y}) = -\alpha \sum_k y_k (\mathbf{e}_k - \mathbf{y})^T \tilde{A}\mathbf{y} e^{\alpha y_k} - \frac{\alpha^2}{2} \sum_k y_k^2 \left(\sigma_k^2 (1-y_k)^2 + \sum_{j \neq k} \sigma_j^2 y_j \right) e^{\alpha y_k}$$
$$- \alpha \int_{\mathbb{R}} \sum_k y_k \left(\sum_j y_j h_j(x) - h_k(x) \right) e^{\alpha y_k} \nu(dx)$$
$$+ \int_{\mathbb{R}} \left[\sum_k \exp\{\alpha y_k\} - \sum_k \exp\left\{ \frac{\alpha y_k \big(1 + h_k(x)\big)}{1 + \sum_j y_j h_j(x)} \right\} \right] \nu(dx).$$

For $\sigma_{\mathbf{min}} := \min\{\sigma_1, \ldots, \sigma_n\}$ and a constant $\beta > 0$ such that $|(\mathbf{e}_k - \mathbf{y})^T \tilde{A}\mathbf{y}| \leq \beta$ for all $\mathbf{y} \in \Delta_n$ and all $k \in \{1, \ldots, n\}$, Imhof showed that

$$-\alpha \sum_k y_k (\mathbf{e}_k - \mathbf{y})^T \tilde{A}\mathbf{y} e^{\alpha y_k} - \frac{\alpha^2}{2} \sum_k y_k^2 \leq \alpha \sum_k y_k e^{\alpha y_k} \left[\beta - \frac{\alpha \sigma_{\mathbf{min}}^2}{2} y_k \big(1 - y_k\big)^2 \right]. \tag{5.2}$$

Furthermore, for $\kappa_{\mathbf{max}} := \sup_{x \in \mathbb{R}} \max\{h_1(x), \ldots, h_n(x)\}$, $\kappa_{\mathbf{min}} := \inf_{x \in \mathbb{R}} \min\{h_1(x), \ldots, h_n(x)\}$, and $M := \int_{\mathbb{R}} \Big(\kappa_{\mathbf{max}} - \kappa_{\mathbf{min}} \Big) \nu(dx)$, we have the inequality

$$\alpha \int_{\mathbb{R}} \sum_k y_k \left(- \sum_j y_j h_j(x) + h_k(x) \right) e^{\alpha y_k} \nu(dx) \leq \alpha \sum_k y_k e^{\alpha y_k} M. \tag{5.3}$$

Recalling the inequality $-e^x \leq -1 - x$ for $x > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\sum_k \exp\{\alpha y_k\} - \sum_k \exp\left\{ \frac{\alpha y_k \big(1 + h_k(x)\big)}{1 + \sum_j y_j h_j(x)} \right\} \right] \nu(dx) \\
&\leq \sum_k \int_{\mathbb{R}} \left[\exp\{\alpha y_k\} - 1 - \frac{\alpha y_k \big(1 + h_k(x)\big)}{1 + \sum_j y_j h_j(x)} \right] \nu(dx) \\
&= \sum_k \int_{\mathbb{R}} \left[\sum_{n=0}^{\infty} \frac{(\alpha y_k)^n}{n!} - 1 - \frac{\alpha y_k \big(1 + h_k(x)\big)}{1 + \sum_j y_j h_j(x)} \right] \nu(dx) \\
&= \sum_k \alpha y_k \int_{\mathbb{R}} \left[\sum_{n=1}^{\infty} \frac{(\alpha y_k)^{n-1}}{n!} - \frac{1 + h_k(x)}{1 + \sum_j y_j h_j(x)} \right] \nu(dx) \\
&\leq \sum_k \alpha y_k \int_{\mathbb{R}} \left[\sum_{n=1}^{\infty} \frac{(\alpha y_k)^{n-1}}{n!} \right] \nu(dx) \\
&= \sum_k \alpha y_k \exp\{\alpha y_k\} \int_{\mathbb{R}} \left[\exp\{-\alpha y_k\} \sum_{n=1}^{\infty} \frac{(\alpha y_k)^{n-1}}{n!} \right] \nu(dx) \\
&\leq \alpha \sum_k y_k \exp\{\alpha y_k\} \nu\big(\mathbb{R}\big).
\end{aligned} \tag{5.4}$$

Collecting Equations (5.2), (5.3), and (5.4), we see that

$$\mathcal{A}_J g(\mathbf{y}) \leq \alpha \sum_k y_k e^{\alpha y_k} \left[\Big(\beta + M + \nu(\mathbb{R}) \Big) - \frac{\alpha \sigma^2_{\mathbf{min}}}{2} y_k \big(1 - y_k\big)^2 \right]$$

Now for an arbitrarily small $\epsilon > 0$, choose $\alpha > 0$ large enough that $\alpha \frac{\sigma^2_{\mathbf{min}}}{2} y\big(1-y\big)^2 \geq \Big(\beta + M + \nu(\mathbb{R})\Big) n + 1$ for all $y \in \left[\frac{1}{n}, 1-\epsilon\right]$. Furthermore, take $\mathbf{y} \in \Delta_n$ such that $y_i \leq 1 - \epsilon$ for all i. For our $\mathbf{y}$, there is at least one y_k such that $y_k \geq \dfrac{1}{n}$ and hence

$$\begin{aligned}
\mathcal{A}_J g(\mathbf{y}) &\leq \alpha \Big(\beta + M + \nu(\mathbb{R}) \Big) \sum_{k:\ y_k < 1/n} y_k e^{\alpha y_k} + \alpha \sum_{k:\ y_k \geq 1/n} y_k e^{\alpha y_k} \Big(- (n-1) \Big(\beta + M + \nu(\mathbb{R}) \Big) - 1 \Big) \\
&\leq \alpha \Big(\beta + M + \nu(\mathbb{R}) \Big)(n-1) \frac{e^{\alpha/n}}{n} + \alpha \frac{e^{\alpha/n}}{n} \Big(- (n-1) \Big(\beta + M + \nu(\mathbb{R}) \Big) - 1 \Big) \\
&= -\alpha \frac{e^{\alpha/n}}{n}.
\end{aligned}$$

Now by Dynkin's formula for every finite T,

$$\begin{aligned}
0 \leq \mathbb{E}_{\mathbf{y}} \Big[g\Big(\mathbf{s}(\tau_\epsilon \wedge T) \Big) \Big] &= g(\mathbf{y}) + \mathbb{E}_{\mathbf{y}} \left[\int_0^{\tau_\epsilon \wedge T} \mathcal{A}_J g\big(\mathbf{s}(t)\big) dt \right] \\
&\leq n e^{\alpha} - \alpha \frac{e^{\alpha/n}}{n} \mathbb{E}_{\mathbf{y}} \Big[\tau_\epsilon \wedge T \Big].
\end{aligned}$$

Therefore, by the monotone convergence theorem, letting $T \to \infty$ yields the inequality $\mathbb{E}_{\mathbf{y}}\big[\tau_\epsilon\big] \leq n^2 \dfrac{e^{\alpha/n}}{n}$.

Finally, take $\epsilon = 1/m$ for $m \in \mathbb{N}$. Then $P_{\mathbf{y}}\left(\sup_{t>0} \max\{s_1(t), \ldots, s_n(t)\} \geq 1 - 1/m \right) = 1$, and therefore

$$1 = P_{\mathbf{y}} \left(\bigcap_{m=1}^{\infty} \left\{ \sup_{t>0} \max\{s_1(t), \ldots, s_n(t)\} \geq 1 - 1/m \right\} \right) = P_{\mathbf{y}} \left(\sup_{t>0} \max\{s_1(t), \ldots, s_n(t)\} = 1 \right).$$

□

Take A to be a payoff matrix for a game. A strategy $\mathbf{p} \in \overline{\Delta}_n$ is called an evolutionary stable strategy if: $\mathbf{q}^T A\mathbf{p} \leq \mathbf{p}^T A\mathbf{p}$ for all $\mathbf{q} \in \overline{\Delta}_n$; and for $\mathbf{q} \in \overline{\Delta}_n$ where $\mathbf{q} \neq \mathbf{p}$ and $\mathbf{q}^T A\mathbf{p} = \mathbf{p}^T A\mathbf{p}$, we have that $\mathbf{q} \cdot A\mathbf{q} < \mathbf{p} \cdot A\mathbf{q}$. Imhof [11] noted that the concept of an evolutionary stable strategy, although a stronger notion than a Nash equilibria, is not strong enough to hold by itself in a stochastic setting. The author adjusted for this weakness by assuming the payoff matrix is conditional negative definite, which is defined below. From these assumptions, the author was then able to show conditions for stability near an evolutionary stable strategy. We utilize this technique developed by the author.

Definition 5.1.1. *A matrix A is said to be conditionally negative definite if for $\mathbf{y} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ where $\mathbf{1}^T\mathbf{y} = 0$, we have*

$$\mathbf{y}^T A \mathbf{y} < 0.$$

Lemma (Imhof [11]). *Suppose that A is an $n \times n$ $(n \geq 2)$ conditionally negative definite matrix, define $\overline{A} = \frac{1}{2}(A + A^T)$ and let λ_2 be the second largest eigenvalue of*

$$D := \overline{A} - \frac{1}{n}\overline{A}\mathbf{1}\mathbf{1}^T - \frac{1}{n}\mathbf{1}\mathbf{1}^T\overline{A} + \frac{\mathbf{1}^T\overline{A}\mathbf{1}}{n}\mathbf{1}\mathbf{1}^T.$$

Then

$$\max_{\substack{\mathbf{x}^T\mathbf{1}=0 \\ \mathbf{x}\neq 0}} \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \lambda_2 < 0.$$

We are now ready to state and prove the theorem.

Theorem 5.1.2. *Take $\mathbf{s}(t)$ as above, $\mathbf{p} \in \Delta_n$ an ESS for our payoff matrix A. Define*

$$\begin{aligned}\kappa_J^2 &= \frac{1}{2}\sum_j p_j\sigma_j^2 - \frac{1}{2\sum_j \sigma_j^{-2}} + \int_B \max_k h_k(x)\nu(dx) \\ &+ \sum_j p_j \int_B \log\left(\frac{1 + \max_k h_k(x)}{1 + h_j(x)}\right)\nu(dx) - \sum_j p_j \int_B h_j(x)\nu(dx)\end{aligned}$$

and assume that $0 < \kappa_J < \frac{n}{n-1}\sqrt{|\lambda_2|}\min_{1\leq j\leq n} p_j$. Furthermore, assume that $\int_B \left(\sum_j (p_j - y_j)h_j(x) - 1\right)\nu(dx) < 0$ for all $\mathbf{y} \in \Delta_n$. Then for $\delta > 0$ such that $\delta^2 > \kappa_J^2/|\lambda_2|$, $y \in \Delta_n$, and $t > 0$, we have the inequalities

$$\mathbb{E}_{\mathbf{y}}\left[\tau_{\overline{U}_\delta(\mathbf{p})}\right] \leq \frac{d(\mathbf{y}, \mathbf{p})}{|\lambda_2|\delta^2 - \kappa_J^2}. \tag{5.5}$$

and

$$\mathbb{E}_{\mathbf{y}}\left[\frac{1}{t}\int_0^t |\mathbf{s}(u) - \mathbf{p}|^2 du\right] \leq \frac{1}{|\lambda_2|}\left(\frac{d\big(\mathbf{y}, \mathbf{p}\big)}{t} + \kappa_J^2\right). \tag{5.6}$$

Lastly, an invariant measure $\pi(\cdot)$ of the stochastic replicator dynamic, exists, is unique, and

$$\pi\Big(U_\delta(\mathbf{p})\Big) \geq 1 - \frac{\kappa_J^2}{|\lambda_2|\delta^2}. \tag{5.7}$$

Proof. Our proof will very closely follow the one given by Imhof [11]. For $\mathbf{p} \in \Delta_n$ an ESS for A, define the function $v(\mathbf{y}) = \sum_j p_j \log\big(p_j/y_j\big)$. Applying $\mathcal{A}_J$ (the infinitesimal generator) to v, we see that

$$\begin{aligned}
\mathcal{A}_J v(\mathbf{y}) = &-\sum_j p_i(\mathbf{e}_i - \mathbf{y})^T[A - \mathbf{diag}(\sigma_1^2, \ldots, \sigma_n^2)]\mathbf{y} + \frac{1}{2}\sum_j p_j\left(\sigma_j^2 - 2y_j\sigma_j^2 + \sum_k y_k^2\sigma_k^2\right) \\
&- \sum_j p_j \int_{\mathbb{R}} \left(h_j(x) - \sum_k y_k h_k(x)\right)\nu(dx) + \sum_j p_j \int_{\mathbb{R}} \log\left(\frac{1 + \sum_k y_k h_k(x)}{1 + h_j(x)}\right)\nu(dx) \\
&= (\mathbf{y} - \mathbf{p})^T A\mathbf{y} - \frac{1}{2}\sum_j y_j^2\sigma_j^2 + \frac{1}{2}\sum_j p_j\sigma_j^2 - \sum_j p_j \int_{\mathbb{R}} \left(h_j(x) - \sum_k y_k h_k(x)\right)\nu(dx) \\
&+ \sum_j p_j \int_{\mathbb{R}} \log\left(\frac{1 + \sum_k y_k h_k(x)}{1 + h_j(x)}\right)\nu(dx) \\
&\le (\mathbf{y} - \mathbf{p})^T A\mathbf{y} - \frac{1}{2}\sum_j y_j^2\sigma_j^2 + \frac{1}{2}\sum_j p_j\sigma_j^2 + \int_{\mathbb{R}} \max_k h_k(x)\nu(dx) - \sum_j p_j \int_{\mathbb{R}} h_j(x)\nu(dx) \\
&+ \sum_j p_j \int_{\mathbb{R}} \log\left(\frac{1 + \max_k h_k(x)}{1 + h_j(x)}\right)\nu(dx).
\end{aligned}$$

In the proof of Theorem 2.1 in [11], Imhof showed that $(\mathbf{y} - \mathbf{p})^T A\mathbf{y} \le \lambda_2|\mathbf{y} - \mathbf{p}|^2$ and $-\dfrac{1}{2}\displaystyle\sum_j y_j^2\sigma_j^2 \le -\dfrac{1}{2\sum_j \sigma_j^{-2}}$. Thus, for $y \in \Delta_n$,

$$\mathcal{A}_J v(\mathbf{y}) \le \lambda_2|\mathbf{y} - \mathbf{p}|^2 + \kappa_J^2.$$

Our assumption $\delta^2 > \kappa_J^2/|\lambda_2|$ tells us for $y \in \Delta_n \backslash U_\delta(\mathbf{p})$, $\mathcal{A}_J v(\mathbf{y}) \le \lambda_2\delta^2 + \kappa_J^2$. By Itô's lemma, the process $v(\mathbf{s}(t)) - (\lambda_2\delta^2 + \kappa_J^2)t$ is a local supermartingale on the interval $[0, \tau_{\overline{U}_\delta(\mathbf{p})})$. Therefore $v(\mathbf{y}) \ge \Big(|\lambda_2|\delta^2 - \kappa_J^2\Big)\mathbb{E}_\mathbf{y}\big[\tau_{\overline{U}_\delta(\mathbf{p})}\big]$, which shows Equation (5.5). The strong Markov property tells us that the stochastic replicator dynamic is recurrent in the set $U_\delta(\mathbf{p})$. Furthermore, by choosing a $\delta_0 > 0$ where $\kappa_J < \delta_0 < \dfrac{n}{n-1}\sqrt{|\lambda_2|}\min\limits_{1 \le j \le n} p_j$, Imhoff [11] in Theorem 2.1 showed that $\overline{\Delta}_n \backslash \Big\{\Delta_n \cap \overline{U}_\delta(\mathbf{p})\Big\} = \emptyset$. Thus, our process never hits the boundary and we are able to pick any $\delta > 0$ for which the inequality holds.

Now define $\tau_k = \inf\{t > 0 : v(\mathbf{s}(t)) = k\}$, where $k > V(\mathbf{y})$. Applying Dynkin's formula we see that

$$\begin{aligned}
0 \le \mathbb{E}_\mathbf{y}\Big[v\big(\mathbf{s}\big(t \wedge \tau_k\big)\Big] &= v(\mathbf{y}) + \mathbb{E}_\mathbf{y}\left[\int_0^{t\wedge\tau_k} \mathcal{A}_J v\Big(\mathbf{s}(u)\Big)du\right] \\
&\le v(\mathbf{y}) + \lambda_2\mathbb{E}_\mathbf{y}\left[\int_0^{t\wedge\tau_k} |\mathbf{s}(u) - \mathbf{p}|^2 du\right] + \kappa_J^2\mathbb{E}_\mathbf{y}\big[t \wedge \tau_k\big]
\end{aligned}$$

Since $t \wedge \tau_k \to t$ as $k \to \infty$, the bounded convergence theorem yields Equation (5.6).

Finally, to show Equation (5.7) we need to show that the transition probabilities converge in total variation to an invariant measure (which makes this measure unique). To accomplish this task we will apply Theorem 5.2 in Down et al [4]. In order to satisfy the hypotheses of the theorem, we need to show that our process is ψ-irreducible (page 1674 [4]) and aperdiodic (page 1675 [4]). To show the ψ-irreducible condition, we define the Borel measure $\psi(O) = M\Big(O \cap U_\delta(\mathbf{p})\Big)$, where M is the Lebesgue measure, and $\eta_O := \displaystyle\int_0^\infty \mathbf{1}_{\{\mathbf{s}(t)\in O\}}dt$ (the occupancy time). Since we know our process is recurrent in $U_\delta(\mathbf{p})$, if $\psi(O) > 0$ then $\mathbb{E}_\mathbf{y}[\eta_O] > 0$.

To show the aperiodic condition we need to find a small Borel set B and a time T such that $P_\mathbf{y}(t, B) > 0$ for all $t \geq T$ and all $\mathbf{y} \in B$. A clear candidate for B is the set $U_\delta(\mathbf{p})$. Before we show this conditions holds, we will note that since the Poisson measure is generated by a Lévy process, (and so the initial condition for Lévy process is Dirac measure δ_0), independent of all the Wiener processes, the jumps are only dependent on time.

To show that this condition holds, we will follow the proof of Claim 1 given in [15]. Since $\nu\big(\mathbb{R}\big) < \infty$, we may rewrite $s(t)$ as

$$\mathbf{s}(t) = \mathbf{y} + \int_0^t \hat{D}^1\big(\mathbf{s}(t-), \mathbf{h}(x)\big)dt + \int_0^t D^2\big(\mathbf{s}(t-)\big)d\mathbf{W}(t) + \int_0^t \int_\mathbb{R} D^3(\mathbf{s}(t-), \mathbf{h}(x))N(dt, dx),$$

where

$$\hat{D}^1\big(\mathbf{y}, \mathbf{h}(x)\big) = \Big[\mathbf{diag}(y_1, \dots, y_n) - \mathbf{y}\mathbf{y}^T\Big]\Big[A - \mathbf{diag}(\sigma_1^2, \dots, \sigma_n^2)\Big]\mathbf{y} + \int_\mathbb{R} \Big(\mathbf{y}\mathbf{h}(x)^T - \mathbf{diag}(h_1(x), \dots, h_n(x))\nu(dx).$$

For the finite interval $[0, t']$ (for any $t' > 0$), there is a positive $P_\mathbf{y}$ probability that a jump does not occur. On this event, $s(t)$ agrees with the process

$$\mathbf{l}(t) = \mathbf{y} + \int_0^t \hat{D}^1\big(\mathbf{l}(t), \mathbf{h}(x)\big)dt + \int_0^t D^2\big(\mathbf{l}(t)\big)d\mathbf{W}(t).$$

Thus, considering Theorem 2.1 in Imhoff [11], the condition holds.

Lastly, we need to show that $\mathcal{A}_J V(\cdot) \leq -cV(\cdot) + b\mathbf{1}_{U_\delta(\mathbf{p})}(\cdot)$ where $V \geq 1$, $V \in D(\mathcal{A}_J)$, and $c, b > 0$. Define $V(\mathbf{y}) = K + \prod_l y_l^{-p_l}$, where K is a positive constant which will later be determined. So

$$\begin{aligned}
\mathcal{A}_J V(\mathbf{y}) = &-\sum_i p_i \left[(\mathbf{e}_i - \mathbf{y})^T \Big[A - \mathbf{diag}(\sigma_1^2, \dots, \sigma_n^2)\Big]\mathbf{y} + \int_\mathbb{R} \Big(\sum_j y_j h_j(x) - h_i(x)\Big)\nu(dx)\right] \prod_l y_l^{-p_l} \\
&+ \frac{1}{2}\sum_i p_i(p_i + 1)\left[(1 - 2y_i)\sigma_i^2 + \sum_j y_j^2\sigma_j^2\right]\prod_l y_l^{-p_l} + \frac{1}{2}\sum_i \sum_{i \neq k} p_i p_k \left[\sum_j y_j^2\sigma_j^2 - y_i\sigma_i^2 - y_k\sigma_k^2\right]\prod_l y_l^{-p_l} \\
&+ \int_\mathbb{R} \Big(V\Big(D^3(\mathbf{y}, \mathbf{h}(x)) + \mathbf{y}\Big) - V\big(\mathbf{y}\big)\Big)\nu(dx) \\
&= (\mathbf{y} - \mathbf{p})^T A\mathbf{y} \cdot \prod_l y_l^{-p_l} + \int_\mathbb{R} \sum_j (p_j - y_j)h_j(x)\nu(dx) \cdot \prod_l y_l^{-p_l} + \sum_j y_j(p_j - y_j)\sigma_j^2 \cdot \prod_l y_l^{-p_l} \\
&+ \sum_i p_i\Big[(1 - 2y_i)\sigma_i^2 + \sum_j y_j^2\sigma_j^2\Big]\prod_l y_l^{-p_l} - \frac{1}{2}\sum_i p_i \sum_{k \neq i} p_k \Big[(1 - 2y_i)\sigma_i^2 + \sum_j y_j^2\sigma_j^2\Big]\prod_l y_l^{-p_l} \\
&+ \frac{1}{2}\sum_i \sum_{i \neq k} p_i p_k \Big[\sum_j y_j^2\sigma_j^2 - y_i\sigma_i^2 - y_k\sigma_k^2\Big]\prod_l y_l^{-p_l} + \int_\mathbb{R} \Big(V\Big(D^3(\mathbf{y}, \mathbf{h}(x)) + \mathbf{y}\Big) - V\big(\mathbf{y}\big)\Big)\nu(dx)
\end{aligned}$$

$$
\begin{aligned}
&= (\mathbf{y}-\mathbf{p})^T A\mathbf{y} \cdot \prod_l y_l^{-p_l} + \int_{\mathbb{R}} \sum_j (p_j - y_j) h_j(x)\nu(dx) \cdot \prod_l y_l^{-p_l} + \sum_j p_i(1-y_j)\sigma_j^2 \cdot \prod_l y_l^{-p_l} \\
&- \frac{1}{2}\sum_i \sum_{k\neq i} p_i p_k \Big[(1-y_i)\sigma_i^2 + y_k \sigma_k^2\Big] \prod_l y_l^{-p_l} + \int_{\mathbb{R}} \Big(V\Big(D^3(\mathbf{y}, \mathbf{h}(x)) + \mathbf{y}\Big) - V\big(\mathbf{y}\big)\Big)\nu(dx) \\
&\leq \Bigg(\lambda_2|\mathbf{p}-\mathbf{y}|^2 + \sum_j p_i(1-y_j)\sigma_j^2 + \int_{\mathbb{R}} \Big(\sum_j (p_j - y_j)h_j(x) - 1\Big)\nu(dx) \\
&- \frac{1}{2}\sum_i \sum_{k\neq i} p_i p_k \Big[(1-y_i)\sigma_i^2 + y_k\sigma_k^2\Big]\Bigg) \prod_l y_l^{-p_l} + \int_{\mathbb{R}} \frac{1+\max_j h_j(x)}{1+\min_j h_j(x)}\nu(dx) := C(\mathbf{y})\prod_l y_l^{-p_l} + \varsigma,
\end{aligned}
$$

for

$$
C(\mathbf{y}) = \lambda_2|\mathbf{p}-\mathbf{y}|^2 + \sum_j p_i(1-y_j)\sigma_j^2 + \int_{\mathbb{R}} \Big(\sum_j (p_j-y_j)h_j(x) - 1\Big)\nu(dx) - \frac{1}{2}\sum_i\sum_{k\neq i} p_i p_k \Big[(1-y_i)\sigma_i^2 + y_k\sigma_k^2\Big]
$$

and

$$
\varsigma = \int_{\mathbb{R}} \frac{1+\max_j h_j(x)}{1+\min_j h_j(x)}\nu(dx).
$$

To finish the inequality, we note that

$$
\begin{aligned}
C(\mathbf{y})\prod_l y_l^{-p_l} + \varsigma &= \left(\frac{C(\mathbf{y})\prod_l y_l^{-p_l}}{V(\mathbf{y})} + \frac{\varsigma}{V(\mathbf{y})}\right)V(\mathbf{y}) \\
&= \left(\frac{C(\mathbf{y})\prod_l y_l^{-p_l}}{K+\prod_l y_l^{-p_l}} + \frac{\varsigma}{K+\prod_l y_l^{-p_l}}\right)V(\mathbf{y}) \leq \Big(C(\mathbf{y}) + \frac{\varsigma}{K}\Big)V\big(\mathbf{y}\big).
\end{aligned}
$$

By our assumptions $C(\mathbf{y}) < 0$ for $\mathbf{y} \in \Delta_n \backslash U_\delta(\mathbf{p})$. Thus, taking K large enough so that $C(\mathbf{y}) + \frac{\varsigma}{K} < 0$ for all $\mathbf{y} \in \Delta_n\backslash U_\delta(\mathbf{p})$ and $V \geq 1$, we are able to find a constants $c, b > 0$ such that $\mathcal{A}_J V(\mathbf{y}) \leq -cV(\mathbf{y}) + b\mathbf{1}_{U_\delta(\mathbf{p})}(\mathbf{y})$ holds for all $\mathbf{y} \in \Delta_n$.

Defining $O^C := \Delta_n \backslash O$ and $\pi(\cdot)$ as the invariant measure, we have

$$
\begin{aligned}
\pi\Big(\overline{U}_\delta(\mathbf{p})^C\Big) &= \lim_{t\to\infty} \mathbb{E}_{\mathbf{y}}\left[\frac{1}{t}\int_0^t \mathbf{1}_{\overline{U}_\delta(\mathbf{p})^C}\big(\mathbf{s}(u)\big)du\right] \\
&\leq \lim_{t\to\infty}\mathbb{E}_{\mathbf{y}}\left[\frac{1}{t}\int_0^t \frac{|\mathbf{s}(u)-\mathbf{p}|^2}{\delta^2}du\right] \leq \frac{\kappa_J^2}{|\lambda_2|\delta^2},
\end{aligned}
$$

and therefore Equation (5.7) follows. □

5.2 Strict Nash and Stochastic Stability in the Presence of Continuous and Random Jumps

We call a strategy $\mathbf{p} \in \overline{\Delta}_n$ a strict Nash Equilibria if for $\mathbf{q} \in \overline{\Delta}_n$ such that $\mathbf{q} \neq \mathbf{p}$, $\mathbf{q}^T A\mathbf{p} < \mathbf{p}^T A\mathbf{p}$. In this section we examine strict Nash Equilibria and how random jumps and white noise affect the stability of replicator dynamics. Throughout this section take pure strategy S_k as a strict Nash Equilibria, i.e., $a_{kk} > a_{jk}$ for all $j \neq k$. Since the characteristics of the jumps are able to impact stability we define the functions $\psi^k_{min}(x) := \min_{j\neq k} h_j(x)$, $\psi^k_{max}(x) := \max_{j\neq k} h_j(x)$, and $\psi_{max}(x) := \max_{j} h_j(x)$. Furthermore, from these functions we define the integrals $I^k_1 := \int_{\mathbb{R}} \frac{\big(h_k(x) - \psi^k_{min}(x)\big)^2}{1+\psi_{max}(x)}\nu(dx)$ and $I^k_2 := \int_{\mathbb{R}} \frac{\psi^k_{min}(x)^2 + h_k - \big(1+h_k(x)\big)\psi^k_{max}(x)}{1+\psi_{max}(x)}\nu(dx)$. Finally, for the purposes of the theorem below, for $\tilde{A}$ defined in Theorem 4.1, define $\beta := \max\{|\tilde{a}_{ji}| : 1 \leq j, i \leq n\}$.

Theorem 5.2.1. *Take the matrix A and the process $\mathbf{s}(t)$ defined in Equation (5.1). Assume that for the pure strategy S_k and the corresponding variance σ^2_k, we have the inequality $a_{kk} > a_{jk} + \sigma^2_k$ for all $j \neq k$ (hence S_k is a strict Nash equilibrium), $h_i(x)$ is nonnegative for all i, and $-2\beta + I^k_2 \geq 0$. Furthermore, for $\alpha > 0$, where $\alpha + a_{jk} < a_{kk} - \sigma^2_k$ for all $j \neq k$, assume $\alpha + 2\beta - I^k_1 \geq 0$. Then for every $\delta > 0$,*

$$P_{\tilde{\mathbf{y}}}\left(\lim_{t\to\infty} \mathbf{s}(t) = \mathbf{e}_k\right) \geq 1 - \frac{1-\tilde{y}_k}{1-\delta}. \tag{5.8}$$

Proof. Take $\tilde{A}$ as in Theorem 4.1. Applying the infinitesimal generator $\mathcal{A}_J$ to our Lyaponuv function $g(\mathbf{y}) = 1 - y_k$, we have

$$\begin{aligned}
\mathcal{A}_J g(\mathbf{y}) &= -y_k(\mathbf{e}_k - \mathbf{y})^T\tilde{A}\mathbf{y} \\
&\quad + -y_k \int_{\mathbb{R}} \left(\frac{1+h_k(x)}{1+\sum_j y_j h_j(x)} + \sum_j y_j h_j(x) - h_k(x) - 1 \right)\nu(dx) \\
&\quad + \int_{\mathbb{R}} \left(1 - \left(\frac{y_k\big(1+h_k(x)\big)}{1+\sum_j y_j h_j(x)} - y_k + y_k \right) + 1 - y_k \right)\nu(dx) \\
&= -y_k(\mathbf{e}_k - \mathbf{y})^T\tilde{A}\mathbf{y} \\
&\quad + -y_k \int_{\mathbb{R}} \left(\frac{h_k(x) - \sum_j y_j h_j(x) + \Big(\sum_j y_j h_j(x)\Big)^2 - h_k(x)\sum_j y_j h_j(x)}{1+\sum_j y_j h_j(x)} \right)\nu(dx)
\end{aligned}$$

Imhof [11] showed that

$$-y_k(\mathbf{e}_k - \mathbf{y})^T\tilde{A}\mathbf{y} \leq -y_k\big[(\alpha+2\beta)y_k - 2\beta\big]g(\mathbf{y}), \tag{5.9}$$

so we must focus on the integral term.

For $\mathbf{y} \in \Delta_n$, we have $1 + \sum_j y_j h_j(x) \geq 1 + \min_j h_j(x) > 0$ by Assumption 2.1. Hence, we may just focus on the numerator of the integrand to find an inequality. Using the $\psi_{\cdot}^k$ functions defined in the beginning of the section, we determine that

$$
\begin{aligned}
&\int_{\mathbb{R}}\left[h_k(x)-\sum_j y_jh_j(x)+\left(\sum_j y_jh_j(x)\right)^2-h_k(x)\sum_j y_jh_j(x)\right]\nu(dx)\\
&=\int_{\mathbb{R}}\left[h_k(x)-y_kh_k(x)-\sum_{j\neq k}y_jh_j(x)+\left(y_kh_k(x)+\sum_{j\neq k}y_jh_j(x)\right)^2-y_kh_k(x)^2-h_k(x)\sum_{j\neq k}y_jh_j(x)\right]\nu(dx)\\
&\geq\int_{\mathbb{R}}\left[h_k(x)-y_kh_k(x)-\psi_{max}^k(x)\sum_{j\neq k}y_j+\left(y_kh_k(x)+\psi_{min}^k(x)\sum_{j\neq k}y_j\right)^2-y_kh_k(x)^2-h_k(x)\psi_{max}^k(x)\sum_{j\neq k}y_j\right]\nu(dx)\\
&=\int_{\mathbb{R}}\left[h_k(x)-y_kh_k(x)-\psi_{max}^k(x)(1-y_k)+\Big(y_kh_k(x)+\psi_{min}^k(x)(1-y_k)\Big)^2-y_kh_k(x)^2-h_k(x)\psi_{max}^k(x)(1-y_k)\right]\nu(dx)\\
&=\int_{\mathbb{R}}\Bigg[h_k(x)-y_kh_k(x)-\psi_{max}^k(x)(1-y_k)+y_k^2h_k(x)^2+2y_kh_k(x)\psi_{min}^k(x)(1-y_k)+\psi_{min}^k(x)^2(1-y_k)^2\\
&-y_kh_k(x)^2-h_k(x)\psi_{max}^k(x)(1-y_k)\Bigg]\nu(dx)\\
&=\int_{\mathbb{R}}\left[h_k-\psi_{max}^k(x)+-y_kh_k(x)^2+2y_kh_k(x)\psi_{min}^k(x)+\psi_{min}^k(x)^2(1-y_k)-h_k(x)\psi_{max}^k(x)\right]\nu(dx)\cdot(1-y_k)\\
&=\int_{\mathbb{R}}\left[-y_k\Big(h_k(x)^2-2h_k(x)\psi_{min}^k(x)+\psi_{min}^k(x)^2\Big)+\psi_{min}^k(x)^2+h_k(x)-\big(1+h_k(x)\big)\psi_{max}^k(x)\right]\nu(dx)\cdot(1-y_k)\\
&=\int_{\mathbb{R}}\left[-y_k\Big(h_k(x)-\psi_{min}^k(x)\Big)^2+\psi_{min}^k(x)^2+h_k(x)-\big(1+h_k(x)\big)\psi_{max}^k(x)\right]\nu(dx)\cdot(1-y_k).
\end{aligned}
$$

Hence we have the inequality

$$
-y_k\int_{\mathbb{R}}\left(\frac{\Big(\sum_j y_jh_j(x)\Big)^2-h_k(x)\sum_j y_jh_j(x)}{1+\sum_j y_jh_j(x)}\right)\nu(dx)\leq -y_k\Big[-I_1^ky_k+I_2^k\Big]g(\mathbf{y}). \tag{5.10}
$$

Thus by Equations (17) and (18)

$$
\mathcal{A}_Jg(\mathbf{y})\leq -y_k\Big[\big(\alpha+2\beta-I_1^k\big)y_k-\big(2\beta-I_2^k\big)\Big]g(\mathbf{y}):=\hat{g}(\mathbf{y}).
$$

With our assumptions we have $\hat{g}(\mathbf{y})\leq 0$ for every $\mathbf{y}\in\Delta_n$. For an arbitrary $\delta>0$, define $V_\delta=\Big\{\mathbf{y}\in\Delta_n : y_k>\delta\Big\}$, and τ_{V_δ} as the first time the process leaves V_δ. Then $g\Big(\mathbf{s}\big(t\wedge\tau_{V_\delta}\big)\Big)$ is a local supermartingale, and thus for $\tilde{\mathbf{y}}\in V_\delta$,

$$
P_{\tilde{\mathbf{y}}}\left(\sup_{0\leq t<\infty}g\Big(\mathbf{s}\big(t\wedge\tau_{V_\delta}\big)\Big)\geq 1-\delta\right)\leq\frac{g\big(\tilde{\mathbf{y}}\big)}{1-\delta}
$$

which implies

$$P_{\tilde{\mathbf{y}}}\left(\sup_{0\le t<\infty} g\Big(\mathbf{s}\big(t\wedge\tau_{V_\delta}\big)\Big) < 1-\delta\right) \ge 1-\frac{g\big(\tilde{\mathbf{y}}\big)}{1-\delta}.$$

Notice that for $\epsilon > 0$, there is a $d > 0$ such that $\hat{g}(\mathbf{y}) \le -d$ on $V_\delta \backslash U_\epsilon(\mathbf{e}_k)$. Therefore, applying the logic given in the proof of Theorem 2 in Kushner [14], we are able to conclude the theorem. □

Remark 5.2.1. *If for all x we have $|\psi_{max}(x) - \psi_{min}(x)|$ is small and $h_k(x)$ is sufficiently larger than $\psi_{max}(x)$, then $I_2^k \ge 0$. Hence, if 2β is small enough, we have $-2\beta + I_k^2 \ge 0$. Furthermore, if $I_1^k < I_2^k$, it is very likely that $2\beta - I_1^k \ge 0$, and so $\alpha + 2\beta - I_1^k \ge 0$. However, if $2\beta - I_1^k < 0$, then α could be large enough so that $\alpha + 2\beta - I_1^k \ge 0$. A case like this is possible with σ_k^2 small and $\min\big\{|a_{kk} - a_{jk}| : 1 \le j \le n \text{ and } j \ne k\big\}$ relatively large.*

Corollary 5.2.1. *Assume that for the pure strategy S_j and the corresponding variance σ_j^2, we have the inequality $a_{kk} > a_{jk} + \sigma_j^2$ for all $j \ne k$, $h_i(x)$ is nonnegative for all i, $-2\beta + I_2^k < 0$, $\alpha + 2\beta - I_1^k > 0$, and $I_1^k \le I_2^k + \alpha/2$. Then there exists a neighborhood of $\mathbf{e}_k$, say $V \subset \Delta_n$, such that for any $\tilde{\mathbf{y}} \in V$ and $\lambda = \sup_{\mathbf{y}\in\partial V} |1 - y_k|$,*

$$P_{\tilde{\mathbf{y}}}\left(\lim_{t\to\infty} \mathbf{s}(t) = \mathbf{e}_k\right) \ge 1 - \frac{1-\tilde{y}_k}{1-\lambda}.$$

Proof. From Theorem 5.1, we have the inequality $\mathcal{A}_J g(\mathbf{y}) \le -y_k\Big[\big(\alpha+2\beta-I_1^k\big)y_k - \big(2\beta - I_2^k\big)\Big]g(\mathbf{y})$ Define $V = \Big\{\mathbf{y} \in \Delta_n : y_k > \dfrac{1}{2}\dfrac{\alpha+4\beta-2I_2^k}{\alpha+2\beta-I_1^k}\Big\}$. Since $-y_k\Big[\big(\alpha+2\beta-I_1^k\big)y_k - \big(2\beta-I_2^k\big)\Big]g(\mathbf{y})$ is negative for $\mathbf{y} \in V$, the rest of the proof follows the proof given in Theorem 5.1.

□

5.3 Dominated Pure Strategies

In the past sections we have shown conditions which the process will converge to a pure strategy but we have not considered the possibility of a pure strategy that the process will never converge to. We say that a strategy $\mathbf{q}$ is dominated by $\mathbf{p}$ if for any strategy you play against your better payoff comes from strategy $\mathbf{p}$, i.e., $\mathbf{q}^T A\mathbf{p}' \le \mathbf{p}^T A\mathbf{p}'$ for all $\mathbf{p}' \in \overline{\Delta}_n$. Our focus will be on the extinction of a dominated pure strategy under appropriate stochastic perturbations.

Theorem 5.3.1. *Let the pure strategy S_k be dominated by the mixed strategy $\mathbf{p} \in \overline{\Delta}_n$. For our payoff matrix A define $K_1 = \min_{\mathbf{q}\in\overline{\Delta}_n} \big\{\mathbf{p}^T A\mathbf{q} - \mathbf{e}_k^T A\mathbf{q}\big\}$, $K_2 = -\dfrac{\sigma_k^2}{2} + \dfrac{1}{2}\sum_j p_j\sigma_j^2$, and define $\sigma_{\max} = \max\{\sigma_1,\ldots,\sigma_n\}$. Suppose that*

$K_2 < K_1$, *and* $h_k(x) \le \sum_j p_j h_j(x)$ *for all* $x \in \mathbb{R}$. *Then for every* $\mathbf{y} \in \overline{\Delta}_n$,

$$P_{\mathbf{y}}\Big(s_k(t) = o\Big(\exp\Big\{t\int_{\mathbb{R}}\Big(h_k(x) - \sum_j p_j h_j(x)\Big)\nu(dx) - (K_1 - K_2)t + 3\sigma_{\max}\sqrt{t\log\log t}\Big\} \Big)\Big) = 1.$$

Proof. We will proceed as in Theorem 3.1 (Imhof [11]). Working from the dominating mixed strategy $\mathbf{p}$, define $G(t) = \log\big(s_k(t)\big) - \sum_j p_j \log\big(s_j(t)\big)$. Itô's lemma yields

$$\begin{aligned} G(t) =& G(0) + \int_0^t \Big(\mathbf{e}_k^T A\mathbf{s}(u) - \mathbf{p}^T A\mathbf{s}(u) - \frac{\sigma_k^2}{2} + \frac{1}{2}\sum_j p_j\sigma_j^2\Big)du \\ &+ \sigma_k W_k(t) - \sum_j p_j \sigma_j W_j(t) \\ &+ t\int_{\mathbb{R}}\Big(\sum_j p_j h_j(x) - h_k(x) + \log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)\Big)\nu(dx) \\ &+ \int_0^t\int_{\mathbb{R}}\log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)\tilde{N}(dx,du). \end{aligned} \tag{5.11}$$

For the integral $\int_0^t\int_{\mathbb{R}}\log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)\tilde{N}(dx,du)$ notice that the integrand is not dependent on the time variable. Hence $\int_0^t\int_{\mathbb{R}}\log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)N(dx,du)$ is a compound Poisson process. Since $\nu\big(\mathbb{R}\big) < \infty$, Theorem 36.5 in Sato [18] tells us that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathbb{R}}\log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)N(dx,du) = \int_{\mathbb{R}}\log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)\nu(dx) \quad \text{a.s.}$$

Therefore

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathbb{R}}\log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)\tilde{N}(dx,du) = 0 \quad \text{a.s.}$$

Now for $\tilde{\sigma} := \Big[(1-p_k)^2\sigma_k^2 + \sum_{j\neq k} p_j^2\sigma_j^2\Big]^{1/2}$, we see that $\tilde{W}(t) := \Big[\sigma_k W_k(t) - \sum_j p_j\sigma_j W_j(t)\Big]/\tilde{\sigma}$ is a standard Wiener process, and that $\tilde{\sigma} \le 3\sigma_{\max}$. Thus, $P_{\mathbf{y}}$ almost surely

$$\begin{aligned} G(t) \le G(0) + \Big(K_1 - K_2\Big)t + \tilde{\sigma}\tilde{W}(t) + \int_0^t\int_{\mathbb{R}}\log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)\tilde{N}(dx,du) \\ + t\int_{\mathbb{R}}\Big(\sum_j p_j h_j(x) - h_k(x) + \log\Big(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\Big)\Big)\nu(dx). \end{aligned} \tag{5.12}$$

Therefore, applying the Law of the Iterated Logarithm, we see that

$$
\begin{aligned}
&\limsup_{t\to\infty} s_k(t)\exp\left[t\int_{\mathbb{R}}\left(h_k(x)-\sum_j p_j h_j(x)\right)\nu(dx)-(K_1-K_2)t-3\sigma_{\max}\sqrt{t\log\log t}\right]\\
&\leq \limsup_{t\to\infty}\exp\left[G(t)+t\int_{\mathbb{R}}\left(h_k(x)-\sum_j p_j h_j(x)\right)\nu(dx)-(K_1-K_2)t-3\sigma_{\max}\sqrt{t\log\log t}\right]\\
&\leq \limsup_{t\to\infty}\exp\left[G(0)+\int_0^t\int_{\mathbb{R}}\log\left(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\right)\tilde{N}(dx,du)+t\int_{\mathbb{R}}\log\left(\frac{1+h_k(x)}{1+\sum_j p_j h_j(x)}\right)\nu(dx)\right.\\
&\qquad\qquad\left.+\,\tilde{\sigma}\tilde{W}(t)-3\sigma_{\max}\sqrt{t\log\log t}\right]=0.
\end{aligned}
\tag{5.13}
$$

$\square$

Chapter 6

Conclusion

6.1 Comparison of the Models for the Two Strategy Case

A natural question arises after the creation of the stochastic models; how close are the stabilities of the stochastic models to the stable equilibria of the deterministic model as well as how the random jumps affect the stability of the Fudenberg and Harris model?

Take

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

as the payoff matrix and $\mathbf{x} = (x_1, x_2)$ to be the replicator dynamics. Recall that the deterministic replicator dynamics simplifies to

$$\dot{x}_1 = x_1(1 - x_1)[(a_{12} - a_{22}) + (a_{11} - a_{21} + a_{22} - a_{12})x_1].$$

With this dynamic, we have the following stability:

I. when $a_{11} - a_{21} > 0$ and $a_{22} - a_{12} < 0$ the only stable equilibrium is the point $(1, 0)$, which is called the strategy 1 dominate game;

II. switching the inequalities, $a_{11} - a_{21} < 0$ and $a_{22} - a_{12} > 0$, the only equilibrium that is stable is the point $(0, 1)$, and is known as the strategy 2 dominate game;

III. when $a_{11} - a_{21} > 0$ and $a_{22} - a_{12} > 0$ the points $(0, 1)$ and $(0, 1)$ are locally asymptotically stable, and $\left(\dfrac{a_{22} - a_{12}}{a_{11} - a_{21} + a_{22} - a_{12}}, \dfrac{a_{22} - a_{12}}{a_{11} - a_{21} + a_{22} - a_{12}}\right)$ is unstable, and since the better payoffs are when both players are playing the same strategy, this is the so called Coordination game;

IV. finally, switching the inequalities above yields $a_{11} - a_{21} < 0$ and $a_{22} - a_{12} < 0$, and we have the endpoints $(1, 0)$ and $(0, 1)$ are unstable, and $\left(\dfrac{a_{12} - a_{22}}{a_{21} - a_{11} + a_{12} - a_{22}}, \dfrac{a_{12} - a_{22}}{a_{21} - a_{11} + a_{12} - a_{22}}\right)$ is stable, which is called the purely mixed strategy game.

We will recall the results of Fudenberg and Harris (§1.3). For an arbitrary $\mathbf{y} \in \Delta_2$, σ_1^2 the variance of the subpopulation playing the first strategy and σ_2^2 the variance of the subpopulation playing the second strategy, Fudenberg and Harris showed that if:

A. $a_{11} - a_{21} > \dfrac{\sigma_1^2 - \sigma_2^2}{2}$ and $a_{22} - a_{12} < \dfrac{\sigma_2^2 - \sigma_1^2}{2}$ then $P_{\mathbf{y}}\left(\lim_{t\to\infty} s(t) = 1\right) = 1$;

B. $a_{11} - a_{21} < \dfrac{\sigma_1^2 - \sigma_2^2}{2}$ and $a_{22} - a_{12} > \dfrac{\sigma_2^2 - \sigma_1^2}{2}$ then $P_{\mathbf{y}}\left(\lim_{t\to\infty} s(t) = 0\right) = 1$;

C. $a_{11} - a_{21} > \dfrac{\sigma_1^2 - \sigma_2^2}{2}$ and $a_{22} - a_{12} > \dfrac{\sigma_2^2 - \sigma_1^2}{2}$ then $P_{\mathbf{y}}\left(\lim_{t\to\infty} s(t) = 0\right) = \dfrac{I_1}{I_1 + I_2}$ and $P_{\mathbf{y}}\left(\lim_{t\to\infty} s(t) = 1\right) = \dfrac{I_2}{I_1 + I_2}$, for I_i defined in the first chapter;

D. $a_{11} - a_{21} < \dfrac{\sigma_1^2 - \sigma_2^2}{2}$ and $a_{22} - a_{12} < \dfrac{\sigma_2^2 - \sigma_1^2}{2}$ then $P_{\mathbf{y}}\left(\sup_{t>0} s(t) = 1\right) = P_{\mathbf{y}}\left(\inf_{t>0} s(t) = 0\right) = 1$.

Clearly, when the difference of the variances is small compared with the difference of the payoffs, the system does not change. For an example, take the strategy 1 dominate game. If $a_{11} - a_{21} = 2$ and $a_{22} - a_{12} = -1$, and the difference of the variances is $\sigma_1^2 - \sigma_2^2 = 3$, then the point $(1, 0)$ is still stable. Although the stable equilibrium in the purely mixed strategy game appears to be negated, in that the stochastic process will not converge to this point, in the long-run there is a large probability that the process will be in a small neighborhood of the deterministic stable point [7].

However, under large differences in the variances, the process could be completely perturbed. Consider when $a_{11} - a_{21} = -1$ and $a_{22} - a_{12} = 2$, and $\sigma_1^2 - \sigma_2^2 = 6$. Deterministically, the population will converge to everyone playing the second strategy, however since the variance of the subpopulation playing the first strategy is large compared to the variance of the subpopulation playing the second strategy and the differences of the payoffs, the population will eventually choose playing the first strategy.

We will now recall the stochastic stability conditions of our approximated evolutionary process. We defined $h_1(x)$ and $h_2(x)$ as the *jump-affect* of the first and second subpopulations respectively, assumed that $h_1(x) - h_2(x) = \epsilon$, where ϵ is for an arbitrary small number, $C_1 := \displaystyle\int_{\mathbb{R}} \frac{1}{1 + h_2(x)} \nu(dx)$, $C_2 := \displaystyle\int_{\mathbb{R}} \frac{1}{\left[1 + h_2(x)\right]^2} \nu(dx)$ If:

2A. $a_{11} - a_{21} > \dfrac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \dfrac{\epsilon^2}{2} C_2$ and $a_{22} - a_{12} < \dfrac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) - \dfrac{\epsilon^2}{2} C_2$ then $P_{\mathbf{y}}\left(\lim_{t\to\infty} \hat{s}(t) = 1\right) = 1$;

2B. $a_{11} - a_{21} < \dfrac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \dfrac{\epsilon^2}{2} C_2$ and $a_{22} - a_{12} > \dfrac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) - \dfrac{\epsilon^2}{2} C_2$ then $P_{\mathbf{y}}\left(\lim_{t\to\infty} \hat{s}(t) = 0\right) = 1$;

2C. $a_{11} - a_{21} > \dfrac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \dfrac{\epsilon^2}{2} C_2$ and $a_{22} - a_{12} > \dfrac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) - \dfrac{\epsilon^2}{2} C_2$ then $P_{\mathbf{y}}\left(\lim_{t\to\infty} \hat{s}(t) = 0\right) = \dfrac{f(1) - f(y_1)}{f(1) - f(0)}$ and $P_{\mathbf{y}}\left(\lim_{t\to\infty} s(t) = 1\right) = \dfrac{f(y_1) - f(0)}{f(1) - f(0)}$;

2D. $a_{11} - a_{21} < \dfrac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \dfrac{\epsilon^2}{2} C_2$ and $a_{22} - a_{12} < \dfrac{\sigma_2^2 - \sigma_1^2}{2} + \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) - \dfrac{\epsilon^2}{2} C_2$ then $P_{\mathbf{y}}\left(\sup_{t>0} \hat{s}(t) = 1\right) = P_{\mathbf{y}}\left(\inf_{t>0} \hat{s}(t) = 0\right) = 1$.

This result is very similar to the Fudenberg and Harris result since we are comparing $a_{11} - a_{21}$ and $a_{22} - a_{12}$ with $\frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \frac{\epsilon^2}{2}C_2$ and $-\left(\frac{\sigma_1^2 - \sigma_2^2}{2} - \epsilon\Big(C_1 - \nu(\mathbb{R})\Big) + \frac{\epsilon^2}{2}C_2\right)$ respectively.

6.2 The General n

We will now consider the theorems for the general n case and how they correspond to the equivalent theorems for the stochastic replicator dynamic diffusion.

The conditions of Theorem 5.1.2 are considerably stronger than the conditions given for the diffusion version of the stochastic replicator dynamic, Theorem 2.1 [11].

For Theorem 5.2.1, the condition that $a_{kk} > a_{jk} + \sigma_j^2$ for all $j \neq k$ tells us that the white noise is not very strong and as such does not command a large presence in the dynamics. For the continuous stochastic replicator dynamic, this is sufficient enough to have convergence to a pure Nash equilibria, given that the initial condition is in the proper neighborhood of the pure strategy, Theorem 4.1 [11]. When considering the right-continuous stochastic replicator dynamic, the added assumptions of $h_i(x)$ is nonnegative for all i, $-2\beta + I_2^k < 0$, $\alpha + 2\beta - I_1^k > 0$, and $I_1^k \leq I_2^k + \alpha/2$, tell us that the size of the jump functions are very close to each other with respect to the payoffs and variances. As such, the jumps are very small perturbations to the dynamics, and the result is very intuitive. However, even with these small perturbations, we are still not able to give an almost sure convergence to the pure strategy. Now, if the jump function h_k is dominant compared to the rest of the jump functions and the payoffs, which is the assumption $-2\beta + I_2^k \geq 0$, then for any neighborhood of the pure strategy, we have a strong probability of converging to the pure strategy. This result revels how commanding the jump perturbations are to the dynamics.

Lastly, Theorem 5.3.1, the conditions are the same with the added $h_k(x) \leq \sum_j p_j h_j(x)$ for every $x \in \mathbb{R}$, which is what we would expect.

6.3 Future Work

Theorem 5.2.1, and Corollary 5.2.1, are not very sharp. I would like to improve these probabilities. Furthermore, the continuous stochastic replicator dynamic has conditions for which the process will converge to a pure strategy almost surely. I am uncertain if conditions can be given so that this characteristic is also true for the right-continuous stochastic replicator dynamic, but I would like to show this to be false or be true.

List of Symbols

S_i - the i^{th} strategy

a_{ij} - The payoff of strategy S_i playing against strategy S_j

A - payoff matrix

u - payoff function

u_i - the fitness function for the i^{th} subpopulation

r_i - the number of players of the i^{th} subpopulation

s_i - the frequency of the i^{th} subpopulation

dt - Lebesgue measure

W_i - standard Wiener process

$dW_i(t)$ - Itó integral

σ_i - variance of the r_i growth

σ - $\sqrt{\sigma_1^2 + \sigma_2^2}$

X - $\mathbb{R}^d$ valued Lévy process

$\Delta X_\omega(s)$ - $X_\omega(s) - X_\omega(s-)$

$N_\omega(t, \cdot)$ - $\#\Big\{0 \leq s \leq t : \Delta X_\omega(s) \in \cdot\Big\}$

$\nu(\cdot)$ - Lévy measure

$\tilde{N}_\omega(dt, dx)$ - $N_\omega(dt, dx) - dt\nu(dx)$

C_b^2 - space of twice differentiable continuous bounded functions

References

[1] Mario Abundo. On first-passage times problem for one-dimensional jump-diffusion processes. *Probability and Mathematical Statistics*, 20(2):399–423, 2000.

[2] D. Applebaum. *Lévy processes and stochastic calculus*. Cambridge Studies in Advanced Mathematics, Cambridge, 2004.

[3] J. Bertoin. *Lévy processes*. Cambridge University Press, Cambridge, 1996.

[4] D. Down, S. P. Meyn, and R. L. Tweedie. Exponential and uniform ergodicity of markov processes. *Annals of Applied Probability*, 23(4):1671–1691, 1995.

[5] E.B. Dynkin. *Markov Processes*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.

[6] B. Fristedt and L. Gray. *A Modern Approach to Probability Theory*. Birkhäuser, Boston, 1997.

[7] D. Fudenberg and C. Harris. Evolutionary dynamics with aggregate shocks. *Journal of Economic Theory*, 57(2):420–441, 1992.

[8] I. Gihman and A. V. Skorohod. *Stochastic differential equations*. Springer-Verlag, New York, 1972.

[9] R. Z. Has'minskiĭ. *Stochastic Stability of Differential Equations*. Sijthoff and Noordhoff, Rockville, Maryland, USA, 2004.

[10] J. Hofbauer and K. Sigmund. *Evolutionary games and population dynamics*. Cambridge University Press, Cambridge, 1998.

[11] I. Imhof. The long-run behavior of the stochastic replicator dynamics. *Annals of Applied Probability*, 15(1B):1019–1045, 2005.

[12] K. Itô and H.P. McKean. *Diffusion Processes and their Sample Paths*. Springer-Verlag, Berlin-Heidelberg-New York, 1965.

[13] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New York, 1991.

[14] H. Kushner. *Stochastic stability and control*. Academic Press Inc., New York, 1967.

[15] H Masuda. Ergodicity and exponential β-mixing bounds for multidimensional diffusions with jumps. *Stochastic Processes and their Applications*, 25(117):35–56, 2007.

[16] J. L. Menaldi. Invariant measure for diffusion with jumps. *Appl Math Optim*, 40(1):105–140, 1999.

[17] S. P. Meyn and R. L. Tweedie. Stability of Markovian Processes III: Foster-Lyapunov criteria for continuous-time processes. *Annals of Applied Probability*, 25(1):518–548, 1993.

[18] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.

[19] A. V. Skorohod. *Asymptotic methods in the theory of stochastic differential equations*. American Mathematical Society, Moscow, 1989.

[20] D. Strook and S.R. Varadhan. *Multidimensional Diffusion Processes*. Springer-Verlag, Berlin Heidelberg New York, 1997.

[21] K. Taira. *Diffusion Processes and Partial Differential Equations*. Academic Press, Inc, San Diego, 1988.

[22] K. Taira. *Semigroups, Boundary Value Problems and Markov Processes.* Springer-Verlag, Berlin-Heidelberg, 2004.

[23] Henry C. Tuckwell. On the first-exit time problem for temporally homogeneous markov processes. *J. Appl. Prob.*, 13(1):39–48, 1976.

[24] P. Young and D. Foster. Cooperation in the short and in the long run. *Games Econ. Behav.*, 3(1):145–156, 1991.